Representation Theory and Cohomology

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Representation theory

G a group and V a vector space

$$G \times V \ \to \ V, \qquad (g,v) \mapsto g \circ v$$

Conditions: $$(g_1 \cdot g_2) \circ v \ = \ g_1 \circ (g_2 \circ v),$$
$$g \circ (a \cdot v + b \cdot w) \ = \ a \cdot (g \circ v) + b \cdot (g \circ w).$$

What sort of groups?

What sorts of vector spaces?

Lie theory

Sophus Lie
1842 - 1899

Perspective of geometry and differential equations

"continuous transformation groups"

acting *continuously*, e.g. on a real or complex vector space V

Lie theory: Understand these continuous representations of G in terms of representations of $\mathfrak{g} = Lie(G)$.

Algebraic groups

Claude Chevalley

1923 - 2003.

Algebraic groups over a field k

GL_n is zero locus inside $\mathbb{A}^{n^2+1}$ of $det(x_{i,j}) \cdot z \ = \ 1$

Fact: Each simple complex Lie group can be viewed as zero locus of further polynomial equations inside some $GL_N(\mathbb{C})$.

Algebraic representations

What is an algebraic representation of an algebraic group?

If V is finite dimensional, $V = k^{\oplus N}$, then $G \times V \to V$ is algebraic (a.k.a. "rational representation") if each matrix coefficient as a function of G is algebraic (i.e., in $k[G]$).

Example: Let $G = GL_2$ act on polynomials of degree n in 2 variables $k[x, y]_n$. Explicitly, we can write this as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ \; x^i y^{n-i} \; = \; (ax + by)^i (cx + dy)^{n-i}$$

Equivalently: comodule structure $\Delta : V \to V \otimes k[G]$, so that $\Delta(v) = \sum v_i \otimes f_i$ with $g \circ v = \sum_i f_i(g) v_i$.

characteristic $p > 0$

Example: $\mathbb{F}_p$ – field of p-elements. (Every non-zero element has an inverse; add 1 to itself p times and the answer is 0.)
Example: $q = p^d$, $d > 0$. There is a unique finite field $\mathbb{F}_q$ of order q.
Example: If F is a field of characteristic $p > 0$, then so is $F(x)$.

If $X \subset \mathbb{A}^N$ is the zero locus of polynomial equations with coefficients in $\mathbb{F}_q$, then sending $(x_1, \ldots, x_N) \in \mathbb{A}^N$ to $(x_1^q, \ldots, x_N^q) \in \mathbb{A}^N$ sends points of X to points of X.

Key point: $(a + b)^p = a^p + b^p$ in characteristic p.

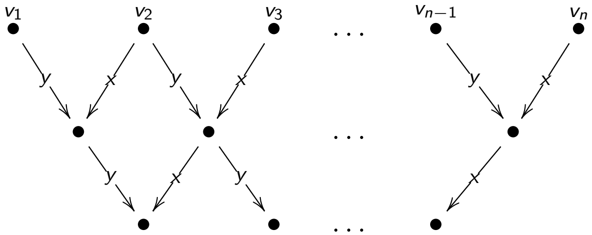
Frobenius map $F^q : X \to X$.

Wildness for finite groups, char $p > 0$

k-linear actions of $\mathbb{Z}/p \times \mathbb{Z}/p$ on k-vector space V correspond to actions of

$$k[g,h]/(g^p = 1 = h^p) \simeq k[x,y]/(x^p,y^p), \quad g = x+1,\ h = y+1$$

Example: Indecomposable, not irreducible



$\mathbb{Z}/p \times \mathbb{Z}/p$ has wild representation type (for $p > 2$).

Solomon Lefschetz

Solomon Lefschetz

1884 - 1972

Applications of algebraic topology to algebraic geometry

(classical) algebraic geometry

Characteristic p STINKS!

Cohomology

Definition

If $G \times M \to M$ is a G-action on the k-vector space M, then

$$H^0(G, M) \;=\; M^G \;=\; \{m \in M; g \circ m = m, \forall g \in G\}.$$

$$H^i(G, M) \;=\; (R^i(H^0(G, -))(M).$$

If every indecomposable G-module is irreducible, then $H^i(G, M) = 0, i > 0$.

$H^1(G, M)$ equals the group of equivalence classes of short exact sequences $0 \to M \to E \to k \to 0$ of G-modules (i.e., extensions of k by M).

Cohomology and Geometry

Daniel Quillen

1940 – 2011

Spectrum of cohomology of a finite group G

$\mathrm{Spec}(H^*(G,k))$, affine algebraic variety

Extension of Quillen by J. Carlson et al: use spirit of Quillen to study representations of a finite group G.

Failure of Lie theory over fields of characteristic $p > 0$

EXAMPLE SL_2 action on the homogeneous polynomials of degree precisely p in two variables: $V = k[x,y]_p$. Inside V, there is a 2-dimensional subrepresentation $W \subset V$ consisting of polynomials linear in x^p, y^p. There is no splitting of $W \subset V$ as representations of SL_2.

V is *indecomposable*, but not *irreducible*.

Much WORSE news:
If action of SL_2 on vector space V factors through $F : SL_2 \to SL_2$, then Lie algebra action is trivial (because differential $d(F) = 0 : \mathfrak{sl}_2 \to \mathfrak{sl}_2$).

Functors and group schemes

Alexander Grothendieck

1928 - 2014

Functorial point of view

A group scheme over k is a *functor*

(comm. k-alg) to (groups).

Replace the Lie algebra of G by "infinitesimal neighborhoods of the identity", so called "Frobenius kernels" $G_{(r)}$.

Frobenius kernels

One can view $G_{(r)} \subset G$ as a representable subfunctor of G

$$(comm.\ k-alg) \to (groups), \quad R \mapsto ker\{F^r : G(R) \to G(R)\}.$$

Example

$GL_{N(r)}$ has coordinate algebra $k[X_{i,j}]/(X_{i,j}^{p^r} - \delta_{i,j})$, a finite dimensional, commutative, local k-algebra; mulitplication of dual $kGL_{N(r)}$ is given by the multiplication of GL_N.

$kG_{(r)}$ is always a f. dim, co-commutative Hopf algebra.

Given a representation $G \times V \to V$, this structure is faithfully reflected by the collection of structures $\{G_{(r)} \times V \to V\}$.

The additive group $\mathbb{G}_a$

Definition

$\mathbb{G}_a$: (comm. k-alg) $\rightarrow$ (abelian groups); $\mathbb{G}_a(R) = R^+$.
$k[\mathbb{G}_a] = k[T]$; group structure determined by comultiplication

$$\Delta : k[T] \rightarrow k[T] \otimes k[T], \quad T \mapsto (T \otimes 1) + (1 \otimes T).$$

Lemma

A $\mathbb{G}_a$-action on a k-vector space V is naturally equivalent to the following data:
An infinite sequence of p-nilpotent, pairwise commuting: operators $u_0, u_1, u_2, u_3 \ldots : V \rightarrow V$ *such that for any* $v \in V$ *all but finitely many* $u_i(v)$ *are 0.*

Varieties for $\mathbb{G}_a$-modules

Definition

The cohomological variety $V^{coh}(\mathbb{G}_a) \equiv Spec_{cont} H^*(\mathbb{G}_a, k)$.

The 1-parameter subgroup variety $V(\mathbb{G}_a) \equiv \{\psi : \mathbb{G}_a \to \mathbb{G}_a\}$.

Proposition

$V^{coh}(\mathbb{G}_a) \simeq \mathbb{A}^\infty \simeq V(\mathbb{G}_a)$.

Definition

$V^{coh}(\mathbb{G}_a)_M \equiv \{\mathfrak{p} \subset H^*(\mathbb{G}_a, k) : \mathfrak{p} \supset ann(H^*(\mathbb{G}_a, M)\}$.

$V(\mathbb{G}_a)_M \equiv \{\psi : \mathbb{G}_a \to \mathbb{G}_a :$
such that NOT all blocks of size p for action at $\psi\}$.

Support varieties for $\mathbb{G}_a$

For M finite dimensional, $V^{coh}(\mathbb{G}_a)_M = V(\mathbb{G}_a)_M \subset \mathbb{A}^\infty$.

- Many "standard" properties including $V(\mathbb{G}_a)_M = \{0\}$ if M is injective, $V(\mathbb{G}_a)_M = \mathbb{A}^\infty$ if $M = k$.
- "Mock injective" modules: there exist (necessarily infinite dimensional) $\mathbb{G}_a$-modules M which are not injective, but $V(\mathbb{G}_a)_M = \{0\}$.
- Know exactly which subvarieties $X \subset \mathbb{A}^\infty$ are of form $X = V(\mathbb{G}_a)_M$ for some finite dimensional $\mathbb{G}_a$-module M.
- Lots of interesting questions about which $X \subset \mathbb{A}^\infty$ are of form $X = V(\mathbb{G}_a)_M$ for an arbitrary $\mathbb{G}_a$-module M.

Other algebraic groups G

Cohomology NOT very useful in general. For example, $H^*(G, k)$ is trivial for G a simple algebraic group.

Will describe a theory using 1-parameter subgroups $\psi : \mathbb{G}_a \to G$ which has many useful properties.

For G unipotent (e.g., $U_N \subset GL_N$), then study of 1-parameter subgroups leads to cohomological calculations.

1-parameter subgroups for infinitesimal kernels $G_{(r)}$

Andrei Suslin

Joint work: Quillen's geometry extends to Frobenius kernels.

computations for $H^*(G_{(r)}, k)$

in terms of the variety of $V(G_{(r)})$ of infinitesimal 1-parameter subgroups $\mathbb{G}_{a(r)} \to G_{(r)}$.

Theorem [Suslin-F-Bendel] $V^{coh}(G_{(r)})_M$ can be identified with the variety $V(G_{(r)})_M$.

Action of G on M at a 1-parameter subgroup

Theorem

Assume that G is a linear algebra group of exponential type. The ind-variety $V(G)$ of 1-parameter subgroups of G is $\simeq$ variety $\mathcal{C}_\infty(\mathcal{N}_p(Lie(G)))$ consisting of finite sequences of p-nilpotent, pair-wise commuting elements of $Lie(G)$:

$$\{\underline{B} \in \mathcal{C}_\infty(\mathcal{N}_p(Lie(G)))\} \stackrel{\sim}{\to} V(G), \quad \underline{B} \mapsto \mathcal{E}_{\underline{B}}.$$

Definition

The action on a rational G-module M at the 1-parameter subgroup $\mathcal{E}_{\underline{B}} : \mathbb{G}_a \to G$ is the action of the p-nilpotent operator

$$\sum_{s \geq 0} (\mathcal{E}_{B_s})_*(u_s) \in kG.$$

Formulation of support variety $V(G)_M$

Definition

The support variety $V(G)_M \subset V(G)$ of M is the subset of those $\underline{B} \in \mathcal{C}_\infty(\mathcal{N}_p(Lie(G)))$ such that $\psi_{\underline{B}}$ has some block of size $< p$.

- For M finite dimensional, $V(G)_M$ carries the same information as the earlier considered $V(G_{(r)})_M$ for $r >> 0$.
- For $G = \mathbb{G}_a$, $V(\mathbb{G}_a)_M \simeq V^{coh}(\mathbb{G}_a)_M$.
- For $G = U_N$, $V^{coh}(U_N)_M$ is much less informative than $V(U_N)_M$.
- Leads to interesting classes of mock injective and mock trivial G-modules.
- Can compute some examples of the form $V(G)_{G/H}$.

"Classical properties" of $M \mapsto V(G)_M$

Theorem

1. *Tensor product:* $V(G)_{M \otimes N} = V(G)_M \cap V(G)_N$.
2. *Two out of three: If* $0 \to M_1 \to M_2 \to M_3 \to 0$, *then the support variety* $V(G)_{M_i}$ *of one of the* M_i *is contained in the union of the support varieties of the other two.*
3. *For the Frobenius twist* $M^{(1)}$ *of* M,

$$V(G)_{M^{(1)}} = \{\mathcal{E}_{(B_0, B_1, B_2 \ldots)} \in V(G) : \mathcal{E}_{(B_1^{(1)}, B_2^{(1)}, \ldots)} \in V(G)_M\}.$$

4. *For any* $r > 0$, *the restriction of* M *to* $kG_{(r)}$ *is injective (equivalently, projective) if and only if the intersection of* $V(G)_M$ *with the subset* $\{\psi_{\underline{B}} : B_s = 0, s > r\} \subset V(G\})$ *equals* $\{\mathcal{E}_{\underline{0}}\}$.

Cohomology for U_N

Strategy: For computations of $H^*(U_{N(r)}, k)$, $H^*(U_N, k)$:

- (F-Suslin) give a means of construction of cohomology classes.
- (Suslin-F-Bendel) give detection of cohomology modulo nilpotents.
- Use the descending central series

$$U_N = \Gamma_1 \supset \Gamma_2 \supset \cdots \subset \Gamma_N = \{e\}$$

  with each subquotient a product of $\mathbb{G}_a$'s.
- Key tool is the T_N-equivariant Lyndon-Hochschild-Serre spectral sequence along with the action of the Steenrod algebra.

LHS Spectral sequence

Key technique for computation is the T_N-equivariant Hochschild-Serre spectral sequence

$$E_2^{*,*} \ = \ H^*(U_N/\Gamma_{v-1},k) \otimes H^*(\Gamma_{v-1}/\Gamma_v,k) \ \Rightarrow \ H^*(U_N/\Gamma_v,k)$$

for terms of the descending central series for U_N.

Compute differentials using the Steenrod algebra: for example, $d_{2p^j+1}^{0,2p^j}((x_{s,t}^{(i)})^{p^j})$ equals

$$\sum_{t=1}^{v-1}(x_{s,s+t}^{(i)})^{p^j} \otimes y_{s+t,s+v}^{(i+1+j)} - (x_{s+t,s+v}^{(i)})^{p^j} \otimes y_{s,s+t}^{(i+1+j)}).$$

We conclude the relation

$$(x_{s,s+1}^{(i)})^{p^{j+1}} \cdot (x_{s_1,s+2})^{(i+1+j)} - (x_{s+1,s+2}^{(i)})^{p^{j+1}} \cdot (x_{s,s+1})^{(i+1+j)} \quad 0 \leq j.$$

Calculations of cohomology

Theorem

For $p \geq N-1$, $H^*((U_N)_{(r)}, k)$ modulo nilpotents is given by explict construction augmenting $k[V_r(U_N)]$.
Similar statement of terms U_N/Γ_v of lower central series.

Remark

This improves [Suslin-F-Bendel] in that we can compare for increasing r, take the limit as r goes to ∞.

Theorem

$\mathrm{Spec}_{cont}\, H^\bullet(U_3, k)$ *is determined by the image of* $H^\bullet(U_3/\Gamma, k) \to H^\bullet(U_3, k)$.

Continuous prime ideal spectrum

Definition

$V^{coh}(G) \equiv \varinjlim_r im\{\text{Spec } H^\bullet(G_{(r)}, k) \to \text{Spec } H^\bullet(G, k)\}$.

Example

$H^*(\mathbb{G}_a, k) = S^*(x^{(i)}, i \geq 1) \otimes \Lambda^*(y^{(i)}, i \geq 0)$, so that $V^{coh}(\mathbb{G}_a) \simeq \mathbb{A}^\infty$.

Proposition

For $p \geq 3$, $H^\bullet(U_N, k)$ embeds in $\varprojlim_r H^\bullet(U_{N(r)}, k)$.

Proposition

There exists a natural, surjective map

$$\text{Proj } V(G)_M \to \text{Proj } V^{coh}(G)_M.$$