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Functional Analysis 2 – Exercise Sheet 8

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Exercise 8.1

Let $P\colon \mathbb{R} \longrightarrow \mathbb{R}$ be the Poisson kernel, given by $P(x) := \frac{1}{\pi}\frac{1}{1+x^2}$ for all $x \in \mathbb{R}$. For $t > 0$ let $P_t(x) := \frac{1}{t}\,P(\frac{x}{t})$ be its L^1-dilation.

a) Show that $\hat{P}(\xi) = \frac{1}{\sqrt{2\pi}} e^{-|\xi|}$ for all $\xi \in \mathbb{R}$.

b) Let $f \in L^2(\mathbb{R})$. Show that the function $u(x,t) := (P_t * f)(x)$ solves the problem

$$\begin{cases} (\partial_t^2 + \partial_x^2)\, u(x,t) = 0 & \text{in } \mathbb{R}\times(0,\infty), \\ u(x,0) = f(x) & \text{for almost all } x \in \mathbb{R}. \end{cases} \tag{1.1}$$

c) Let $T\colon [0,\infty) \longrightarrow L^2(\mathbb{R})$ be given by $T(0) = \mathsf{id}$ and $T(t)f := P_t * f$ for all $t > 0$. Show that T is a C^0–semigroup on $L^2(\mathbb{R})$.

<u>Proof:</u> We denote $X := L^2(\mathbb{R})$.

a) Let $\xi \in \mathbb{R}$ and $g(\xi) = \frac{1}{\sqrt{2\pi}} e^{-|\xi|}$, then we have

$$2\pi\,\hat{g}(x) = \int_{\mathbb{R}} e^{-|\xi| - \mathbf{i}\,x\xi}\,\mathrm{d}\xi = \int_0^\infty e^{-\xi-\mathbf{i}\,x\xi} + e^{-\xi+\mathbf{i}\,x\xi}\,\mathrm{d}\xi \tag{1.2}$$

$$= -\frac{1}{1+\mathbf{i}x}\Big[e^{-\xi-\mathbf{i}\,x\xi}\Big]_0^\infty + \frac{1}{1-\mathbf{i}x}\Big[e^{-\xi+\mathbf{i}\,x\xi}\Big]_0^\infty \tag{1.3}$$

$$= \frac{1}{1+\mathbf{i}x} + \frac{1}{1-\mathbf{i}x} = \frac{1-\mathbf{i}x+1+\mathbf{i}x}{1+x^2} = 2\pi P(x). \tag{1.4}$$

Since both functions g and P are radially symmetric, we get from Fourier inversion that $\hat{P} = g$.

b) First note that by Young's convolution inequality the convolution is well defined and $u \in X$. Also note from the dilation theorem of Fourier transforms we get $\hat{P}_t(\xi) = g(t\xi)$ for all $t > 0$ and $\xi \in \mathbb{R}$. So we get from the product formula of Fourier transforms (in the x variable) $\hat{u}(t,\xi) = \sqrt{2\pi}\,\hat{P}_t(\xi)\,\hat{f}(\xi)$, from which we deduce

$$\mathcal{F}[(\partial_t^2 + \partial_x^2)\,u(\,\cdot\,,t)] = \partial_t^2\hat{u}(t,\xi) - \xi^2\hat{u}(t,\xi) = -\xi^2 e^{-t|\xi|}\,\hat{f}(\xi) + \xi^2 e^{-t|\xi|}\,\hat{f}(\xi) = 0. \tag{1.5}$$

Fourier inversion then yields, that u is a solution to the differential equation. The boundary data is correct since the family $(P_t)_t$ is a Dirac sequence.

c) For $t, s > 0$ we see

$$P_t * P_s = \mathcal{F}^{-1}[\mathcal{F}[P_t * P_s]] = \sqrt{2\pi}\mathcal{F}^{-1}[e^{-(t+s)|\,\cdot\,|}] = P_{t+s}. \tag{1.6}$$

In b) we already clarified that $P_t * f \to f$ in X (since $(P_t)_t$ is a Dirac sequence). This makes T into a C^0–semigroup. ∎

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Exercise 8.2
Let X be a Banach space and let T be a C^0–semigroup. For every $t > 0$ let $T(t)$ be invertible and $T(t)^{-1} \in \mathcal{L}(X)$.

a) Show that $S\colon [0,\infty) \longrightarrow \mathcal{L}(X)$ defined by $S(t) := T(t)^{-1}$ is a C^0-semigroup.

b) Let A be the generator of T. Show that $-A$ is the generator of S.

c) Define $U\colon \mathbb{R} \longrightarrow \mathcal{L}(X)$ via

$$U(t) := \begin{cases} T(t) & \text{if } t \geq 0, \\ S(-t) & \text{if } t < 0. \end{cases} \tag{2.1}$$

Show that U is a C^0–group.

Proof:

a) Since $T(0) = \mathsf{id}$ we have $S(0) = \mathsf{id}$. For $s, t > 0$ we have

$$S(t+s) = T(t+s)^{-1} = T(s+t)^{-1} = (T(s)T(t))^{-1} = T(t)^{-1}T(s)^{-1} = S(t)S(s). \tag{2.2}$$

It remains to show the strong continuity. Note that $T(t)$ is bijective for every $t \geq 0$. Fix $s > 1$ and let $x \in X$, then there exists $y \in X$ such that $T(s)y = x$. Then we have for every $t < 1$

$$\|S(t)x - x\| = \|S(t)T(t)T(s-t)y - T(s)y\| = \|T(s-t)y - T(s)y\| \xrightarrow{t\to 0} 0 \tag{2.3}$$

by the continuity of T (c.f. Lemma 4.3). Altogether, this makes S into a C^0–semigroup.

b) Let $x \in \mathcal{D}(A)$. Since S is a C^0–semigroup by a), there exist $M \geq 1$ and $\omega \geq 0$ such that $\|S(t)\|_{op} \leq Me^{\omega t}$. Let $t > 0$, then it holds

$$\left\|\frac{S(t)x - x}{t} - (-Ax)\right\| = \left\|S(t)\left[\frac{x - T(t)x}{t}\right] - (-Ax)\right\| \tag{2.4}$$

$$\leq \|S(t)\|_{op}\left\|\left[\frac{x - T(t)x}{t}\right] - T(t)(-Ax)\right\| \tag{2.5}$$

$$\leq \|S(t)\|_{op}\left[\left\|\frac{x - T(t)x}{t} - (-Ax)\right\| + \|T(t)(-Ax) - (-Ax)\|\right] \tag{2.6}$$

$$\leq Me^{\omega t}\left[\left\|\frac{x - T(t)x}{t} - (-Ax)\right\| + \|T(t)(-Ax) - (-Ax)\|\right]. \tag{2.7}$$

In the limit $t \to 0$ we obtain that $-A$ is the generator of S. Note that $\mathcal{D}(A) = \mathcal{D}(-A)$.

c) Since $T(0) = S(0) = \mathsf{id}$ we have $U(0) = \mathsf{id}$. By the strong continuity of both S and T, we also know that U is strongly continuous, i.e. $U(t)x \to x$ for $t \to 0$. It remains to show the semigroup property. For this purpose, let $s, t \in \mathbb{R}$.

If $s > 0, t > 0$, or $s < 0, t < 0$, then the statement is clear. Let $s < 0$ and $t > 0$. Assume first $t > |s|$. Then it holds

$$U(t)U(s) = T(t)S(-s) = T(t+s)T(-s)S(-s) = T(t+s) = U(t+s). \tag{2.8}$$

If $t < |s|$, then we have

$$U(t)U(s) = T(t)S(-s) = T(t)S(t)S(-t-s) = S(-(t+s)) = U(t+s). \tag{2.9}$$

So altogether, U is a C^0–group. ■

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Exercise 8.3
Let $T\colon [0,\infty) \longrightarrow \mathcal{L}(L^2(\mathbb{R}))$ be defined by $T(t)f := f(t + \cdot\,)$ for all $t \geq 0$.

a) Show that T is a C^0–semigroup of contractions on $L^2(\mathbb{R})$.

b) Show that the operator $Af := f'$ on the domain $\mathcal{D}(A) := H^1(\mathbb{R})$ is the generator of T.

c) Show the inequality of Landau–Kolmogorov: For every $f \in H^2(\mathbb{R})$ it holds

$$\|f'\|^2_{L^2(\mathbb{R})} \leq 4\,\|f\|_{L^2(\mathbb{R})}\,\|f''\|_{L^2(\mathbb{R})}. \tag{3.1}$$

<u>**Proof:**</u> Let $X := L^2(\mathbb{R})$.

a) Obviously we have $T(0) = \mathsf{id}$ and for $t, s > 0$ we have

$$T(t)T(s)f(x) = T(t)f(t + \cdot\,)(x) = f(t + s + x) = T(t + s)f(x). \tag{3.2}$$

It remains to show the strong continuity. For this purpose, we first show the continuity on the dense subspace $C_c^\infty(\mathbb{R})$. Let $f \in C_c^\infty(\mathbb{R})$, then we have by the fundamental lemma of calculus, Jensen's inequality, and Fubini's theorem

$$\|f - T(t)f\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} |f(x) - f(x + t)|^2 \,\mathrm{d}x = \int_{\mathbb{R}} \Big| \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} f(x + (1 - s)t)\,\mathrm{d}s \Big|^2 \,\mathrm{d}x \tag{3.3}$$

$$\leq t^2 \int_{\mathbb{R}} \int_0^1 |f'(x + (1 - s)t)|^2 \,\mathrm{d}s\,\mathrm{d}x = t^2 \|f'\|^2_{L^2}. \tag{3.4}$$

In the limit $t \to 0$ we have $T(t)f \to f$ in X. Now for arbitrary $f \in X$, we can find $g \in C_c^\infty(\mathbb{R})$ which approximates f in X. Then we have

$$\|f - T(t)f\|_{L^2(\mathbb{R})} \leq \|f - g\|_{L^2(\mathbb{R})} + \|g - T(t)g\|_{L^2(\mathbb{R})} + \|T(t)g - T(t)f\|_{L^2(\mathbb{R})}. \tag{3.5}$$

Using a 3ε argument shows the strong continuity of T. Altogether, T is a C^0–semigroup. Since $\|T(t)f\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}$, we have $\|T(t)\|_{op} = 1$ for all $t \geq 0$, so T is a C^0–semigroup of contractions.

b) We again focus first on functions $f \in C_c^\infty(\mathbb{R})$. We know that

$$\frac{T(t)f(x) - f(x)}{t} \xrightarrow{t \to 0} f'(x) \qquad \text{for all } x \in \mathbb{R} \tag{3.6}$$

pointwise. Since f and f' are continuous, the difference quotient is bounded on a compact set and by Lebesgue's dominated convergence theorem we have

$$\frac{T(t)f - f}{t} \xrightarrow{t \to 0} f' \qquad \text{in } X. \tag{3.7}$$

We now take the closure of $C_c^\infty(\mathbb{R})$ with respect to the L^2–norm, such that the derivative also is in $L^2(\mathbb{R})$, which is the Sobolev space $H^1(\mathbb{R})$.

c) Using the same arguments as in b), we have $\mathcal{D}(A^2) = H^2(\mathbb{R})$. The statement then follows from the fact, that T is semigroup of contractions and Lemma 4.9 (with $M = 1$).

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Exercise 8.4
Let X be a Banach space. Let T be a C^0–semigroup and let $A\colon \mathcal{D}(A) \longrightarrow X$ be its generator. Show Taylor's formula

$$T(t)x = x + tAx + \int_0^t (t-s)\,T(s)A^2x\,\mathrm{d}s \qquad \text{for all } x \in \mathcal{D}(A^2). \tag{3.1}$$

Proof: Let $x \in \mathcal{D}(A^2)$, then also $x \in \mathcal{D}(A)$. Define the function $h\colon [0,\infty) \longrightarrow X$ by $h(t) \coloneqq T(t)x$. We already know from Theorem 4.5 that the map h is differentiable and the derivative is given by $h'(t) = T(t)Ax$. Since $x \in \mathcal{D}(A^2)$ we can again take a derivative, which results in $h''(t) = T(t)A^2x$. Using Taylors formula with integral remainder gives

$$h(t) = h(0) + t\,h'(0) + \int_0^t (t-s)\,h''(s)\,\mathrm{d}s = x + tAx + \int_0^t (t-s)\,T(s)A^2x\,\mathrm{d}s, \tag{3.2}$$

which was the claim. ■