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Functional Analysis 2 – Exercise Sheet 7

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Exercise 7.1
Let $H = L^2(\mathbb{R}^n, \mathbb{C})$ and $\mathcal{F} \colon H \longrightarrow H$ be the Fourier transform. Show that $\sigma_p(\mathcal{F}) = \{\pm 1, \pm \mathbf{i}\}$.

Hint: Apply the Fourier transform several times to the eigenvalue equation. Compute the Fourier transforms of functions of the form $x\, e^{-x^2}$, $(1+x^2)\, e^{-x^2}$ and $x\,(1+x^2)\, e^{-x^2}$.

Proof: The eigenvalue equation reads $\mathcal{F}u = \lambda u$ for some $u \in H$ and $\lambda \in \mathbb{C}$. Applying $\mathcal{F}$ three times again results in $u = \lambda^4\, u$ and therefore $\sigma_p \subset \{\pm 1, \pm \mathbf{i}\}$. It remains to find an eigenvector for each potential eigenvalue.

Define $\mathcal{G}(x) \coloneqq e^{-\frac{1}{2}|x|^2}$, then we know that $\mathcal{F}\mathcal{G} = \mathcal{G}$ and thus $1 \in \sigma_p$. We also find for $j \in \{1, \dots, n\}$

$$\partial_j \mathcal{G}(x) = -x_j\, \mathcal{G}(x), \qquad \partial_j^2 \mathcal{G}(x) = (x_j^2 - 1)\, \mathcal{G}(x), \qquad \partial_j^3 \mathcal{G}(x) = (3x_j - x_j^3)\, \mathcal{G}(x). \tag{1.1}$$

We can use this information to construct eigenvectors. We get

$$\mathcal{F}[x_j\, \mathcal{G}(x)](\xi) = \mathbf{i}\, \partial_j \mathcal{F}\mathcal{G}(\xi) = -\mathbf{i}\, \xi_j\, \mathcal{G}(\xi) \qquad \Longrightarrow -\mathbf{i} \in \sigma_p(\mathcal{F}), \tag{1.2}$$

$$\mathcal{F}[(-2 + x_j^2)\mathcal{G}(x)](\xi) = -2\mathcal{F}\mathcal{G}(\xi) - \partial_j^2 \mathcal{F}\mathcal{G}(\xi) = -(-2 + \xi_j^2)\mathcal{G}(\xi) \qquad \Longrightarrow -1 \in \sigma_p(\mathcal{F}), \tag{1.3}$$

$$\mathcal{F}[(2x_j - x_j^3)\mathcal{G}(x)](\xi) = \mathbf{i}\,(2\xi_j + \xi_j^3)\, \mathcal{G}(\xi) \qquad \Longrightarrow \mathbf{i} \in \sigma_p(\mathcal{F}). \tag{1.4}$$

Altogether we have shown $\sigma_p = \{\pm 1, \pm \mathbf{i}\}$. ■

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Exercise 7.2
Let $j \in \{1, \dots, n\}$ and let $f \in L^2(\mathbb{R}^n, \mathbb{C})$. Show that there exists a unique solution $u \in H^1(\mathbb{R}^n, \mathbb{C})$ such that $(\mathbf{i} \pm \mathbf{i}\partial_j)u = f$ almost everywhere in $\mathbb{R}^n$.

Proof: Apply the Fourier transform to both sides to find

$$\hat{u}(\xi) = \frac{\hat{f}(\xi)}{\mathbf{i} \pm \xi_j}. \tag{2.1}$$

Since $\xi \in \mathbb{R}^n$, the right hand side will never be zero in the denominator and remains an L^2-function by Hölders inequality. Fourier inversion then yields the unique solution. ■

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Exercise 7.3
Let X be a Banach space and $T \in \mathcal{L}(X)$. Let $f\colon \mathbb{R}^n \longrightarrow X$ be Bochner integrable. Show that Tf is a Bochner integrable function and that

$$\int_{\mathbb{R}^n} Tf(x)\,\mathrm{d}x = T\Big(\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x\Big). \tag{3.1}$$

Hint: A quick repetition of the Bochner integration theory: A function $g\colon \mathbb{R}^n \longrightarrow X$ is called *simple* if there exist finitely many Lebesgue measurable sets $\{A_j\}_j$ with $|A_j| < \infty$ and elements $\{\alpha_j\}_j \subset X$, such that $g = \sum_j \alpha_j \chi_{A_j}$. For simple functions we define the Bochner integral

$$\int_{\mathbb{R}^n} g(x)\,\mathrm{d}x \coloneqq \sum\nolimits_j \alpha_j\,|A_j|. \tag{3.2}$$

A function $h\colon \mathbb{R}^n \longrightarrow X$ is called *Bochner measurable* if there exists a sequence $(g_k)_k$ of simple functions, such that $g_k \to h$ pointwise almost everywhere. A Bochner measurable function $f\colon \mathbb{R}^n \longrightarrow X$ is called *Bochner integrable* if there exists a sequence of simple functions $(g_k)_k$, such that $g_k \to f$ almost everywhere and

$$\int_{\mathbb{R}^n} \|g_i(x) - g_j(x)\|\,\mathrm{d}x \to 0 \qquad \text{as } i, j \to \infty. \tag{3.3}$$

We then define the Bochner integral of f via

$$\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x \coloneqq \lim_{k\to\infty} \int_{\mathbb{R}^n} g_k(x)\,\mathrm{d}x. \tag{3.4}$$

In order to show the statement: consider first simple functions and then go to the limit.

Proof: Let us first show the statement for simple functions, then we know that there exist finitely many measurable sets $\{A_j\}_j$ with $|A_j| < \infty$ and $\{\alpha_j\}_j \subset X$ such that $f = \sum_j \alpha_j \chi_{A_j}$. Then we have $Tf = \sum_j T\alpha_j \chi_{A_j}$ and it holds

$$\int_{\mathbb{R}^n} Tf(x)\,\mathrm{d}x = \sum_j T\alpha_j\,|A_j| = T\sum_j \alpha_j\,|A_j| = T\Big(\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x\Big). \tag{3.5}$$

Now, let f be Bochner integrable and let $(f_j)_j$ be a sequence of simple functions, such that $f_j \to f$ almost everywhere and $\int_{\mathbb{R}^n} \|f_i(x) - f_j(x)\|\,\mathrm{d}x \to 0$ as i, $j \to \infty$. Then $(Tf_j)_j$ is a sequence of simple function and we have

$$\int_{\mathbb{R}^n} \|Tf_i(x) - Tf_j(x)\|\,\mathrm{d}x \le \|T\|_{\mathrm{op}} \int_{\mathbb{R}^n} \|f_i(x) - f_j(x)\|\,\mathrm{d}x \to 0 \qquad \text{as } i, j \to \infty. \tag{3.6}$$

By continuity of T, it also holds $Tf_j \to Tf$ almost everywhere, so Tf is Bochner integrable and it holds

$$\int_{\mathbb{R}^n} Tf(x)\,\mathrm{d}x = \lim_{j\to\infty} \int_{\mathbb{R}^n} Tf_j(x)\,\mathrm{d}x \overset{(3.5)}{=} T\Big(\lim_{j\to\infty} \int_{\mathbb{R}^n} f_j(x)\,\mathrm{d}x\Big) = T\Big(\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x\Big). \tag{3.7}$$

This was the claim. ■