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Functional Analysis 2 – Exercise Sheet 6

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Exercise 6.1

let H be a separable Hilbertspace and $A \in \mathcal{L}(H)$ a self–adjoint operator. Let $\{\mu_n\}_n$ be the (possibly inifinite) collection of spectral measures with respect to A. Show that

$$\sigma(A) = \overline{\bigcup_n \operatorname{spt} \mu_n}. \tag{1.1}$$

<u>**Proof:**</u> By definition of the spectral measure we have $\operatorname{spt} \mu_n \subset \sigma(A)$ for all n, and therefore

$$\overline{\bigcup_n \operatorname{spt} \mu_n} \subset \sigma(A), \tag{1.2}$$

since $\sigma(A)$ is compact (c.f. Theorem 1.10). So it suffices to show the converse inclusion. Note that we only need to show the statement for the multiplication operator by the unitary invariance of spectral measures (c.f. Lemma 3.19) and the spectral theorem (c.f. Theorem 3.15). Assume that there exists $\alpha \in \sigma(A) \setminus \overline{\bigcup_n \operatorname{spt} \mu_n}$. Since this set is open in $\sigma(A)$, we can find $\varepsilon > 0$, such that $\Omega \coloneqq B_\varepsilon(\alpha) \subset \sigma(A) \setminus \overline{\bigcup_n \operatorname{spt} \mu_n}$. Let μ_n be the spectral measure associated with $\psi_n \in L^2(\mathbb{R}, \mathbb{C})$ for all n. Since A is the multiplication operator, we can find a non–negative function $f \in C^0(\sigma(A))$ such that $\operatorname{spt}(f) \subset \Omega$ and therefore

$$0 = \int_{\sigma(A)} f(\lambda)\, \mathrm{d}\mu_n(\lambda) = (\psi_n, f(\lambda)\psi_n) = \int_{\sigma(A)} f(\lambda)\, |\psi_n(\lambda)|^2 \,\mathrm{d}\lambda, \tag{1.3}$$

from which we deduce $\psi_n = 0$ for all n, a contradiction. Therefore the claim holds. ■

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Exercise 6.2
Let $H := L^2((0,1),\mathbb{C})$ and let $h \in H$. Let $\lambda \in \mathbb{C}$. In this exercise we want to solve the Sturm–Liouville problem

$$\begin{cases} u''(x) + \lambda u(x) = h(x) & \text{for almost all } x \in (0,1), \quad (2.1a) \\ u(0) = u(1) = 0. & (2.1b) \end{cases}$$

Using methods of the theory of ordinary differential equations we know that the problem (2.1) is equivalent to the integral equation $u - \lambda\, Ku = -Kh$, where $K\colon H \longrightarrow H$ is an integral operator defined by

$$Kf(x) := \int_0^1 k(x,t)\, f(t)\,\mathrm{d}t, \qquad k(x,t) := \min\{x,t\} - xt. \tag{2.2}$$

a) Determine the spectrum of K.

b) Give necessary and sufficient conditions on λ so that (2.1) has a unique solution.

c) Give a representation formula for the solution to (2.1) in terms of h and λ.

Proof:

a) Since $k \in L^2((0,1)\times(0,1),\mathbb{R})$ we know that K is a Hilbert–Schmidt operator and therefore a compact operator $K\colon H \longrightarrow H$. Therefore it is clear that $0 \in \sigma(K)$. Moreover, since k is symmetric and real–valued, K is a self–adjoint operator, which means that $\sigma(K) = \{0\} \cup \sigma_p(K)$. In order to determine the eigenvalues, let $\lambda \neq 0$. By the self–adjointness of K we know that $\lambda \in \mathbb{R}$. Then we are looking for non–trivial solutions $v \in H$ of the equation

$$Kv - \lambda v = 0. \tag{2.3}$$

Note that the kernel k is weakly differentiable and therefore we have Kv is in the Soboloev space $H^1((0,1),\mathbb{C})$, so that v itselft must be in the Sobolev space $H^1((0,1),\mathbb{C})$. Boot–strapping shows that $v \in C^\infty((0,1),\mathbb{C})$. We can explicitely compute

$$\begin{aligned} Kv(x) &= \int_0^x (t - xt)\, v(t)\,\mathrm{d}t + \int_x^1 (x - xt)\, v(t)\,\mathrm{d}t && (2.4) \\ &= (1-x)\int_0^x t\, v(t)\,\mathrm{d}t + x\int_x^1 (1-t)\, v(t)\,\mathrm{d}t \overset{!}{=} \lambda v(x). && (2.5) \end{aligned}$$

Taking the derivative with respect to x on both sides results in

$$-\int_0^x t\, v(t)\,\mathrm{d}t + (1-x)\, x\, v(x) + \int_x^1 (1-t)\, v(t)\,\mathrm{d}t - x\,(1-x)\, v(x) = \lambda\, v'(x), \tag{2.6}$$

from which follows that

$$\int_x^1 (1-t)\, v(t)\,\mathrm{d}t - \int_0^x t\, v(t)\,\mathrm{d}t = \lambda\, v'(x). \tag{2.7}$$

Taking the derivative again results in

$$-(1-x)\, v(x) - x\, v(x) = \lambda v''(x) \iff v''(x) + \frac{1}{\lambda} v(x) = 0. \tag{2.8}$$

Solutions to this equation are given by

$$v(x) = a\, \sin(\tfrac{1}{\sqrt{\lambda}} x) + b\, \cos(\tfrac{1}{\sqrt{\lambda}} x), \tag{2.9}$$

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where $a, b, c \in \mathbb{C}$. Matching the boundary conditions (2.1b) we need to have $\frac{1}{\sqrt{\lambda}} = k\pi$ for some $k \in \mathbb{Z}$, which means that $\lambda_k := \frac{1}{k^2\pi^2}$ is an eigenvalue for every $k \in \mathbb{Z}$.

So in conclusion we have

$$\sigma(K) = \{\lambda_k : k \in \mathbb{Z}\} \cup \{0\}. \tag{2.10}$$

b) Using the spectral theorem we know that we can extract an orthogonal basis for H of the eigenvectors of K. Let $\psi_k(x) := c_k \sin(\lambda_k x)$ for $c_k \neq 0$ and $k \in \mathbb{Z}^* := \mathbb{Z} \setminus \{0\}$. The factors c_k are chosen such that $\|\psi_k\| = 1$ for all $k \neq 0$. Inserting this information in (2.1a) leads to

$$\sum_{k\in\mathbb{Z}^*} -\frac{1}{\lambda_k} (u, \psi_k)\, \psi_k + \lambda \sum_{k\in\mathbb{Z}^*} (u, \psi_k)\, \psi_k = \sum_{k\in\mathbb{Z}^*} (h, \psi_k)\, \psi_k. \tag{2.11}$$

Reshuffling the terms yields

$$\sum_{k\in\mathbb{Z}^*} (\lambda - k^2\pi^2)\, (u, \psi_k)\, \psi_k = \sum_{k\in\mathbb{Z}^*} (h, \psi_k)\, \psi_k. \tag{2.12}$$

This means that there exists a solution if and only if $\lambda \neq k^2\pi^2$ for all $k \in \mathbb{Z}^*$.

c) Let $\lambda \neq k^2\pi^2$ for all $k \in \mathbb{Z}$ by b). Testing the equation $u - \lambda\, Ku = -Kh$ with ψ_k results in

$$(1 - \lambda\, \lambda_k)\, (u, \psi_k) = -\lambda_k\, (h, \psi_k) \qquad \text{for all } k \in \mathbb{Z}^*, \tag{2.13}$$

which leads to

$$(u, \psi_k) = \frac{1}{\lambda - k^2\pi^2}\, (h, \psi_k) \qquad \text{for all } k \in \mathbb{Z}^*. \tag{2.14}$$

So the solution to (2.1) is given by

$$u(x) = \sum_{k\in\mathbb{Z}^*} \frac{(h, \psi_k)}{\lambda - k^2\pi^2}\, \psi_k \qquad \text{for all } x \in (0, 1). \tag{2.15}$$

We therefore have found a representation of the solution in terms of λ and h. ■

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Exercise 6.3
Let X be a Banach space, $\mathcal{D}(A) \subset X$ a dense subspace and $A\colon \mathcal{D}(A) \longrightarrow X$ a linear operator. Let $\rho(A) \neq \emptyset$. Show that A is a closed operator.

Proof: Let $\lambda \in \rho(A)$. Let $(x_k)_k \subset \mathcal{D}(A)$ be a convergent sequence with limit $x \in X$ and let $y \in X$, such that $Ax_k \to y$. We need to show, that $x \in \mathcal{D}(A)$ and that $Ax = y$. Let $R \coloneqq R_\lambda(A) = (\lambda \,\mathsf{id} - A)^{-1}\colon X \longrightarrow \mathcal{D}(A)$ be the resolvent of A with respect to λ. Note that the resolvent is a continuous operator. Then it holds

$$x \xleftarrow{k\to\infty} x_k = R(\lambda \,\mathsf{id} - A)x_k = \lambda\, Rx_k - RAx_k \xrightarrow{k\to\infty} \lambda\, Rx - Ry. \tag{3.1}$$

Since $\operatorname{ran}(R) = \mathcal{D}(A)$, it holds $x = R(\lambda x - y) \in \mathcal{D}(A)$. Applying R^{-1} to this equations yields

$$\lambda x - Ax = \lambda x - y \quad \Longrightarrow \quad y = Ax. \tag{3.2}$$

Therefore the operator A is closed. ■