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Functional Analysis 2 – Exercise Sheet 5

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Exercise 5.1
Let H be a separable Hilbert space and $A \in \mathcal{J}_2(H)$. Show that A is a compact operator.

Hint: Find an approximation of finite–rank operators.

Proof: Let $(\psi_k)_k \subset H$ be a orthonormal basis, then we know that

$$\operatorname{Tr}(|A|^2) = \sum_{k\in\mathbb{N}} \|A\psi_k\|^2 < \infty. \tag{1.1}$$

For $N \in \mathbb{N}$ we define now the operator $A_N \in \mathcal{L}(H)$ via

$$A_N x := \sum_{k=1}^{N} (\psi_k, x)\, A\psi_k \qquad \text{for all } x \in H. \tag{1.2}$$

The operator A_N is a finite rank operator for all $n \in \mathbb{N}$ and therefore compact. Moreover we have for every $x \in H$

$$\begin{aligned} \|Ax - A_N x\| &\leq \sum_{k=N+1}^{\infty} |(\psi_k, x)|\, \|A\psi_k\| && (1.3)\\ &\leq \Big(\sum_{k=N+1}^{\infty} |(\psi_k, x)|^2 \Big)^{\frac{1}{2}} \Big(\sum_{k=N+1}^{\infty} \|A\psi_k\|^2 \Big)^{\frac{1}{2}} && (1.4)\\ &\leq \|x\| \Big(\sum_{k=N+1}^{\infty} \|A\psi_k\|^2 \Big)^{\frac{1}{2}}. && (1.5) \end{aligned}$$

In total we get

$$\|A - A_N\|_{op} \leq \Big(\sum_{k=N+1}^{\infty} \|A\psi_k\|^2 \Big)^{\frac{1}{2}} \xrightarrow{N\to\infty} 0. \tag{1.6}$$

So we found an approximation of A of finite rank operators. Using Proposition 1.26 we obtain that A is a compact operator. ∎

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Exercise 5.2
Let X be a Banach space and let $K \in \mathcal{L}(X)$ be a compact operator. Let $U \subset X$ be open, such that $0 \in U$ and let $N\colon U \longrightarrow X$ such that

$$\frac{\|Nx\|}{\|x\|} \longrightarrow 0 \qquad \text{for } \|x\| \to 0. \tag{2.1}$$

Assume there exists a $\lambda \in \mathbb{K} \setminus \{0\}$ and a sequence $(\lambda_k)_k \subset \mathbb{K}$ and $(x_k)_k \subset U$ with the following properties:

a) $x_k \neq 0$ for all $k \in \mathbb{N}$ and $x_k \to 0$ as $k \to \infty$.

b) $\lambda_k \neq \lambda$ for all $k \in \mathbb{N}$ and $\lambda_k \to \lambda$ as $k \to \infty$.

c) $\lambda_k x_k = Kx_k + Nx_k$ for all $k \in \mathbb{N}$.

Show that λ is an eigenvalue of K.

Hint: Assume that λ is not an eigenvalue and use theorems of Fredholm operators to obtain a suitable resolvent. Find a representation for x_k in terms of that resolvent and lead this to a contradiction.

Proof: Assume λ is not an eigenvalue of K. Since K is a compact operator we know from Theorem 2.3 that $\mathsf{id} - \frac{1}{\lambda}K$ is a Fredholm operator and since λ is not an eigenvalue we know that $\mathsf{id} - \frac{1}{\lambda}K$ is injective. Since the index is 0 we know that this operator is also surjective and therefore the inverse $R \coloneqq (\mathsf{id} - \frac{1}{\lambda}K)^{-1} \in \mathcal{L}(X)$. We obtain

$$x_k = \Big(\frac{1}{\lambda_k} - \frac{1}{\lambda}\Big)RKx_k + \frac{1}{\lambda_k}RNx_k. \tag{2.2}$$

We take the modulus and divide by $\|x_k\| \neq 0$ and obtain

$$1 \lesssim \Big|\frac{1}{\lambda_k} - \frac{1}{\lambda}\Big| + \frac{\|Nx_k\|}{\|x_k\|} \longrightarrow 0 \qquad \text{for } k \to \infty. \tag{2.3}$$

A contradiction. So λ is an eigenvalue of K. ∎

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Exercise 5.3

Let H be a separable Hilbert space and let $A \in \mathcal{L}(H)$ be a compact and self–adjoint operator. Let $a^* \geq 0$ be the biggest eigenvalue of A and $a_* \leq 0$ be the smallest eigenvalue of A.

a) Show that one of the two equalities $a^* = \|A\|$ or $a_* = -\|A\|$ holds.

b) Let $B \in \mathcal{L}(H)$ be another compact and self–adjoint operator. Let $b^* \geq 0$ be the biggest eigenvalue of B and $b_* \leq 0$ be the smallest eigenvalue of B. Let $\lambda^* \geq 0$ be the biggest eigenvalue of $A + B$ and $\lambda_* \leq 0$ be the smallest eigenvalue of $A + B$. Show that

$$\lambda^* \leq a^* + b^*, \qquad \lambda_* \geq a_* + b_*. \tag{3.1}$$

Proof:

a) From the spectral theorem we know that $\sigma(A) \setminus \{0\} \subset \sigma_p(A)$. From Theorem 3.1 we know that $A = 0$ if $a^* = a_* = 0$, so without loss of generality we can assume that a^* or a_* are not trivial. We also know that

$$a_* = \inf_{\|x\|=1} (x, Ax), \qquad a^* = \sup_{\|x\|=1} (x, Ax). \tag{3.2}$$

Using Exercise 3.3 we know that

$$\max\{a^*, |a_*|\} = \sup_{\lambda \in \sigma(A)} |\lambda| = \sup_{\|x\|=1} |(x, Ax)| = \|A\|, \tag{3.3}$$

from which the claim follows.

b) If $A + B = 0$ then the statement is trivial. Otherwise we get from Theorem 3.1

$$\lambda_* = \inf_{\|x\|=1} (x, Ax) + (x, Bx) \geq a_* + b_*, \tag{3.4}$$

and similarly

$$\lambda^* = \sup_{\|x\|=1} (x, Ax) + (x, Bx) \leq a^* + b^*. \tag{3.5}$$

So the claim holds. ■

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Exercise 5.4
Let $\mathcal{H} := L^2((0,1),\mathbb{C})$ and $\mathcal{D}(A) := \{f \in H^2((0,1),\mathbb{C}) : f(0) = f(1),\ f'(0) = f'(1)\}$. Let $A\colon \mathcal{D}(A) \longrightarrow \mathcal{H}$ be the periodic Laplace operator (see Example 1.21). Determine all eigenvalues and eigenfunctions of A. Do these eigenfunctions form an orthonormal basis of $\mathcal{H}$? Justify your answer.

Proof: We need to find solutions of the following boundary value problem

$$\begin{cases} u''(x) = \lambda\, u(x) & \text{for all } x \in (0,1), \quad (4.1a) \\ u(0) = u(1), & (4.1b) \\ u'(0) = u'(1), & (4.1c) \end{cases}$$

where $\lambda \in \mathbb{C}$ takes the role of an eigenvalue. Using Example 1.21, we know that $\lambda \in \mathbb{R}_+$ since A is a symmetric operator and positive. Solutions to (4.1a) are given by

$$u(x) = a\,\sin(\sqrt{\lambda}\,x) + b\,\cos(\sqrt{\lambda}\,x) + k \qquad \text{for all } x \in (0,1) \tag{4.2}$$

for parameters $a, b, k \in \mathbb{C}$. In order to match the boundary conditions (4.1b) and (4.1c) we need to choose a and b appropriately. For this prupose we define

$$s := \sin(\sqrt{\lambda}), \qquad c := \cos(\sqrt{\lambda}). \tag{4.3}$$

Assuming $s \neq 0$ or equivalently $c \neq 1$, which corresponds to $\lambda = 4\pi^2 j^2$ for some $j \in \mathbb{N}$, from (4.1b) we get

$$b \overset{!}{=} as + bc \quad \Longleftrightarrow \quad b = a\,\frac{s}{1-c}, \tag{4.4}$$

and from (4.1c) we get

$$\sqrt{\lambda}a \overset{!}{=} \sqrt{\lambda}\,ac - \sqrt{\lambda}\,bs \quad \Longleftrightarrow \quad b = a\,\frac{c-1}{s}. \tag{4.5}$$

Combining (4.4) and (4.5) we obtain

$$\begin{aligned} \frac{c-1}{s} = \frac{s}{1-c} \quad &\Longleftrightarrow \quad -(1-c)^2 = s^2 && (4.6)\\ &\Longleftrightarrow \quad -1 + 2c = 1 && (4.7)\\ &\Longleftrightarrow \quad c = 1 && (4.8)\\ &\Longleftrightarrow \quad \lambda = 4\pi^2 j^2 \qquad \text{for } j \in \mathbb{Z}. && (4.9) \end{aligned}$$

So in total, we get the eigenvalues $\lambda_j = 4\pi^2 j^2$ for all $j \in \mathbb{Z}$ with corresponding eigenfunctions

$$u_j(x) = a\,\sin(2\pi j x) + b\,\cos(2\pi j x) + k \qquad \text{for all } x \in (0,1), \tag{4.10}$$

where $a, b, k \in \mathbb{C}$ are arbitrary parameters (which means that every eigenspace has dimension 3, except for $j = 0$, for which the dimension is 1).

Setting $a = \mathbf{i}$, $b = 1$ and $k = 0$, we obtain for every $j \in \mathbb{Z}$ the eigenfunction $u_j(x) = e^{2\pi \mathbf{i} j x}$. From analysis, we know that $\operatorname{span}\{x \longmapsto e^{2\pi\mathbf{i}jx}\}_{j\in\mathbb{Z}}$ is dense in $\mathcal{H}$ (Fourier series), so we can extract an orthonormal basis. ∎

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Exercise 5.5
Let $\mathcal{H} \coloneqq L^2((0,1),\mathbb{C})$. Define the operator

$$Au(x) \coloneqq \int_0^x u(y)\,\mathrm{d}y \qquad \text{for all } u \in \mathcal{H}. \tag{5.1}$$

a) Show that $A\colon \mathcal{H} \longrightarrow \mathcal{H}$ is a compact operator.

b) Determine $\sigma_p(A)$ and $\sigma(A)$.

c) Is A a self–adjoint operator? Justify your answer.

Hint: You can use compact embedding theorems from the theory of Sobolev spaces.

Proof:

a) We first want to show, that $A \in \mathcal{L}(\mathcal{H})$. Linearity is trivial, so it remains to show continuity. Let $u \in \mathcal{H}$, then we have with the help of Jensen's inequality

$$\|Au\|^2 = \int_0^1 \left| \int_0^x u(y)\,\mathrm{d}y \right|^2 \mathrm{d}x \le \int_0^1 x \int_0^x |u(y)|^2\,\mathrm{d}y\,\mathrm{d}x \le \|u\|^2. \tag{5.2}$$

So $A\colon \mathcal{H} \longrightarrow \mathcal{H}$ is well–defined and continuous.

We now claim that $v \coloneqq Au \in H^1((0,1),\mathbb{C})$ and $v' = u$. Indeed, by density we find a sequence $(u_k)_k \subset C_c^\infty((0,1),\mathbb{C})$, such that $u_k \to u$ in $\mathcal{H}$. Since A is continuous, we find that $v_k \coloneqq Au_k \to v$. By the fundamental theorem of calculus we have $(v_k)_k \subset C^\infty((0,1),\mathbb{C})$. Let $\varphi \in C_c^\infty((0,1),\mathbb{C})$, then we have by the fundamental theorem of calculus

$$\int_0^1 v_k(x)\,\varphi'(x)\,\mathrm{d}x = -\int_0^1 v_k'(x)\,\varphi(x)\,\mathrm{d}x = -\int_0^1 u_k(x)\,\varphi(x)\,\mathrm{d}x. \tag{5.3}$$

In the limit we find that u is the weak derivative of v. Since $u, v \in \mathcal{H}$, we have $v \in H^1((0,1),\mathbb{C})$.

Using the Sobolev embedding theorem, we know that $H^1((0,1),\mathbb{C})$ is compactly embedded in $\mathcal{H}$, so A is a compact operator.

b) We first want to determine $\sigma_p(A)$. Let $\lambda \in \mathbb{C}$. In the case $\lambda = 0$ we are looking for a solution $u \in \mathcal{H}$ such that

$$\int_0^x u(y)\,\mathrm{d}y = 0 \qquad \text{for almost all } x \in (0,1). \tag{5.4}$$

This can only be the case if $u = 0$ almost everywhere. So $\ker(A) = \{0\}$ and thus $0 \notin \sigma_p(A)$.

Now let $\lambda \neq 0$ take the role of a potential eigenvalue, so we are looking for a solution

$$\int_0^x u(y)\,\mathrm{d}y = \lambda\, u(x) \qquad \text{for almost all } x \in (0,1). \tag{5.5}$$

Keep in mind that from this condition it follows that $u \in H^1((0,1),\mathbb{C}) \subset C^0((0,1),\mathbb{C})$ since the left hand side is in the Sobolev space (see computations from a)). So boundary values are well defined. An equivalent formulation of (5.5) is

$$\begin{cases} u'(x) = \frac{1}{\lambda}\, u(x) & \text{for almost all } x \in (0,1), \qquad (5.6a) \\ u(0) = 0. & \qquad (5.6b) \end{cases}$$

The only solutions to this equation is given by $u = 0$ almost everywhere. So $\ker(A - \lambda \,\mathsf{id}) = \{0\}$ and it follows that $\sigma_p(A) = \emptyset$.

Since A is a compact operator we already know that $0 \in \sigma(A)$. We claim that $\sigma(A) = \{0\}$ and in order to show this, we compute the spectral radius of the operator. Using Exercise 1.4 we know that

$$\sup_{\lambda \in \sigma(A)} |\lambda| = \lim_{k \to \infty} \|A^k\|^{\frac{1}{k}}, \tag{5.7}$$

so we compute A^k for all $k \in \mathbb{N}$. Via induction[1] we get

$$A^k u(x) = \frac{1}{(k-1)!} \int_0^x (x-y)^{k-1} u(y) \,\mathrm{d}y \tag{5.8}$$

and similar to the continuity of A we obtain $\|A\|_{op} \leq \frac{1}{(k-1)!}$. From this we deduce that

$$\sup_{\lambda \in \sigma(A)} |\lambda| = \lim_{k \to \infty} \|A^k\|^{\frac{1}{k}} = 0, \tag{5.9}$$

and thus $\sigma(A) = \{0\}$.

c) A is not self–adjoint because otherwise it would have an eigenvalue (see Theorem 3.1). ■

[1] The induction requires the integration by parts formula, but similar to (5.3) we obtain that we can use it.

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Exercise 5.6
Let H be a separable Hilbert space and $A \in \mathcal{L}(H)$ be self–adjoint. Let $N \in \mathbb{N} \cup \{\infty\}$. Show that there exists a decomposition

$$H = \bigoplus_{n=1}^{N} H_n \tag{6.1}$$

with subspaces $H_n \subset H$ for every $n \in \{1, \dots, N\}$, such that:

a) For every $n \in \{1, \dots, N\}$ the space H_n is A invariant, i.e. for $x \in H_n$ we have $Ax \in H_n$.

b) For every $n \in \{1, \dots, N\}$ there exists $y_n \in H_n$, such that y_n is cyclic for the restriction $A|_{H_n}$, i.e.

$$H_n = \overline{\{f(A)y_n : f \in C^0(\sigma(A))\}}. \tag{6.2}$$

<u>**Proof:**</u> Let $\{\mathbf{e}_k\}_{k\in\mathbb{N}} \subset H$ be an orthonormal basis (possible by separability of H). Define the spaces

$$E_n \coloneqq \operatorname{span}\{\mathbf{e}_1, \dots, \mathbf{e}_n\} \qquad \text{for all } n \in \mathbb{N}. \tag{6.3}$$