Denis Brazke
denis.brazke@uni-heidelberg.de

Functional Analysis 2 – Exercise Sheet 4

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Exercise 4.1

Let H be a separable Hilbertspace and $A \in \mathcal{J}_1$. Show that the series $\sum_k (\varphi_k, A\varphi_k)$ converges absolutely for every orthonormal basis $(\varphi_k)_k \subset H$ and that the limit is independent of the choice of the orthonormal basis.

Hint: For the independence you can use that every operator $A \in \mathcal{J}_1$ is decomposable into $A = A_+ - A_-$, where $A_+, A_- \geq 0$ and $A_+ A_- = 0$.

Proof: Since $A \in \mathcal{J}_1$ we can find an partial isometry U such that $A = U\,|A|^{\frac{1}{2}}\,|A|^{\frac{1}{2}}$. From this representation we see that

$$|(\varphi_k, A\varphi_k)| \leq \|\,|A|^{\frac{1}{2}}\,U^*\varphi_k\|\;\|\,|A|^{\frac{1}{2}}\varphi_k\| \qquad \text{for all } k \in \mathbb{N}. \tag{1.1}$$

With the inequality of Cauchy–Schwarz we obtain

$$\sum_{k\in\mathbb{N}} |(\varphi_k, A\varphi_k)| \leq \Big(\sum_{k\in\mathbb{N}} \|\,|A|^{\frac{1}{2}}\,U^*\varphi_k\|^2\Big)^{\frac{1}{2}} \Big(\sum_{k\in\mathbb{N}} \|\,|A|^{\frac{1}{2}}\,\varphi_k\|^2\Big)^{\frac{1}{2}}. \tag{1.2}$$

Both factors are finite since we know from Theorem 1.32

$$\sum_{k\in\mathbb{N}} \|\,|A|^{\frac{1}{2}}U^*\varphi_k\|^2 = \sum_{k\in\mathbb{N}} (\varphi_k, U^*|A|\,U\varphi_k) \leq \operatorname{Tr}|A| = \|A\|_{\mathcal{J}_1} < \infty, \tag{1.3}$$

$$\sum_{k\in\mathbb{N}} \|\,|A|^{\frac{1}{2}}\varphi_k\|^2 = \sum_{k\in\mathbb{N}} (\varphi_k, |A|\varphi_k) \leq \operatorname{Tr}|A| = \|A\|_{\mathcal{J}_1} < \infty. \tag{1.4}$$

Thus the series converges absolutely. To show independence, let $(\psi_j)_j$ be another orthonormal basis of H. We decompose $A = A_+ - A_-$ with $A_+, A_- \geq 0$ and $A_+A_- = 0$ (see spectral theorem later). So it suffices to show that the sum is independent for $A \geq 0$. Using Parseval's identity we get

$$\sum_{k\in\mathbb{N}} (\varphi_k, A\varphi_k) = \sum_{k\in\mathbb{N}} \|A^{\frac{1}{2}}\varphi_k\|^2 = \sum_{k\in\mathbb{N}}\sum_{j\in\mathbb{N}} |(\psi_j, A^{\frac{1}{2}}\varphi_k)|^2 \tag{1.5}$$

$$= \sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{N}} |(A^{\frac{1}{2}}\psi_j, \varphi_k)|^2 = \sum_{j\in\mathbb{N}} \|A^{\frac{1}{2}}\psi_j\|^2 = \sum_{j\in\mathbb{N}} (\psi_k, A\psi_k). \tag{1.6}$$

So the series in independent of the orthonormal basis. ■

Denis Brazke
denis.brazke@uni-heidelberg.de

Exercise 4.2
Let X and Y be Banach spcaes and $A \in \mathcal{L}(X, Y)$ be a Fredholm operator. Show that the adjoint operator $A' \in \mathcal{L}(Y', X')$ is a Fredholm operator and $\operatorname{ind}(A') = -\operatorname{ind}(A)$.

Proof: We find closed subspaces $X_0 \subset X$ and $Y_0 \subset Y$, such that

$$X = X_0 \oplus \ker(A), \qquad Y = \operatorname{ran}(A) \oplus Y_0, \tag{2.1}$$

with their duals

$$X' = X_0' \oplus \ker(A)', \qquad Y' = \operatorname{ran}(A)' \oplus Y_0'. \tag{2.2}$$

Since A is a Fredholm operator, the spaces $\ker(A)$ and Y_0 are finite dimensional and therefore also their dual spaces $\ker(A)'$ and Y_0' with the same dimensions. From this, the index formula follows easily. It remains to show that $\operatorname{ran}(A')$ is closed. Notice that $\operatorname{ran}(A') = X_0'$.

Let $J\colon X_0 \longrightarrow X$ and $R\colon Y \longrightarrow \operatorname{ran}(A)$ be the inclusion and restriction map respectively from Proposition 2.6. Then the operator $B := JAR\colon X_0 \longrightarrow \operatorname{ran}(A)$ is a continuous bijection and since $\operatorname{ran}(A)$ is closed it follows that $B^{-1}\colon \operatorname{ran}(A) \longrightarrow X_0$ is a continuous linear map. Since inverse of adjoints are the adjoints of the inverse, we obtain that $B'\colon \operatorname{ran}(A)' \longrightarrow X_0'$ is a continuous bijective map and therefore maps closed spaces onto closed spaces. So the space $\operatorname{ran}(A') = X_0'$ is a closed space. Altogether the claim follows. ■

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Exercise 4.3
Let X, Y and Z be Banach spaces and let $B \in \mathcal{L}(X,Y)$ and $A \in \mathcal{L}(Y,Z)$. Show that if two of the three operators A, B and AB are Fredholm operators, then also the third one is a Fredholm operator and it holds

$$\operatorname{ind}(AB) = \operatorname{ind}(A) + \operatorname{ind}(B). \tag{3.1}$$

In order to do so, show the following statements:

a) Show that $\dim\ker(AB) = \dim\ker(B) + \dim(\ker(A) \cap \operatorname{ran}(B))$.

b) Decompose the spaces Y, $\ker(A)$ and $\operatorname{ran}(B)$ suitably and show, that Z decomposes in $\operatorname{ran}(AB)$, a subspace of $\operatorname{ran}(A)$ and a closed subspace $Z_0 \subset Z$.

c) Show that $\operatorname{codim}\operatorname{ran}(AB) = \operatorname{codim}\operatorname{ran}(A) + \operatorname{codim}\operatorname{ran}(B) - \dim\ker(A) + \dim(\operatorname{ran}(B) \cap \ker(A))$.

d) Show the closedness of the ranges.

e) Conclude the statement.

Hint: Considering b): use more than one decomposition of the space Y.

This exercise will be longer than the others since there are a lot of cases to consider, so the amount of points will be doubled for this exercise. Apart from d) and e) this is an exercise in linear algebra.

Proof:

a) Let $X_0 \subset X$ a subspace, such that $X = \ker(B) \oplus X_0$. Then the restriction $B|_{X_0}$ is injective and $\operatorname{ran}(B|_{X_0}) = \operatorname{ran}(B)$ and it holds

$$B^{-1}(V) = \ker(B) \oplus B|_{X_0}^{-1}(V) \qquad \text{for all subspaces } V \subset Y. \tag{3.2}$$

We therefore get

$$\ker(AB) = B^{-1}(\ker(A) \cap \operatorname{ran}(B)) = \ker(B) \oplus B|_{X_0}^{-1}(\ker(A) \cap \operatorname{ran}(B)). \tag{3.3}$$

We get from this equality

$$\dim\ker(AB) = \dim\ker(B) + \dim(\ker(A) \cap \operatorname{ran}(B)). \tag{3.4}$$

Note, that if A and B are Fredholm operators, then the right hand side is finite and therefore the left. If AB is a Fredholm operator, then the right hand side must be finite.

b) We find spaces Y_0, Y_1, Y_2 and Y_3 so that

$$Y = \operatorname{ran}(B) \oplus Y_0, \qquad \ker(A) = (\operatorname{ran}(B) \cap \ker(A)) \oplus Y_2, \tag{3.5}$$

$$Y = (\operatorname{ran}(B) + \ker(A)) \oplus Y_1, \qquad \operatorname{ran}(B) = Y_3 \oplus (\operatorname{ran}(B) \cap \ker(A)). \tag{3.6}$$

With these decompositions we have $\operatorname{ran}(B) + \ker(A) = \operatorname{ran}(B) \oplus Y_2$ and thus

$$Y = \operatorname{ran}(B) \oplus Y_2 \oplus Y_1 = Y_3 \oplus (\operatorname{ran}(B) \cap \ker(A)) \oplus Y_2 \oplus Y_1 = Y_3 \oplus \ker(A) \oplus Y_1, \tag{3.7}$$

in particular $Y_0 \simeq Y_1 \oplus Y_2$. So the restriction $A|_{Y_3 \oplus Y_1} : Y_3 \oplus Y_1 \longrightarrow \operatorname{ran}(A)$ is a bijection and we have

$$\operatorname{ran}(A) = AY_3 \oplus AY_1 = A(Y_3 \oplus \ker(A)) \oplus AY_1 = \operatorname{ran}(AB) \oplus AY_1. \tag{3.8}$$

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Note that by the bijectivity of the restriction we get $AY_1 \simeq Y_1$. Since we can find a subspace $Z_0 \subset Z$ and a decomposition $Z = \operatorname{ran}(A) \oplus Z_0$, we get

$$Z = \operatorname{ran}(AB) \oplus AY_1 \oplus Z_0, \tag{3.9}$$

which was the claim.

c) Note that by the bijectivity of the restriction of A regarded in b) we get $AY_1 \simeq Y_1$. Assuming first that all three operators A, B and AB are Fredholm, we get from b)

$$\begin{aligned}
\operatorname{codim}\operatorname{ran}(AB) &= \operatorname{codim}\operatorname{ran}(A) + \dim(Y_1) &(3.10)\\
&= \operatorname{codim}\operatorname{ran}(A) + \dim(Y_0) - \dim(Y_2) &(3.11)\\
&= \operatorname{codim}\operatorname{ran}(A) + \operatorname{codim}\operatorname{ran}(B) - \dim Y_2 &(3.12)\\
&= \operatorname{codim}\operatorname{ran}(A) + \operatorname{codim}\operatorname{ran}(B) - \dim\ker(A) + \dim(\operatorname{ran}(B)\cap\ker(A)). &(3.13)
\end{aligned}$$

Note that we already know from a) that $\dim(\operatorname{ran}(B)\cap\ker(A))$ is finite. It remains to show that this is indeed well-defined.

- If A and B are Fredholm operators, then we know Y_0 is finite dimensional and therefore Y_1 and Y_2 by the aformentioned isomorphy. Also the quantities $\operatorname{codim}\operatorname{ran}(A)$ and $\dim\ker(A)$ are finite, so that the derivation (3.10) – (3.13) is well-defined and the claim holds.
- Assuming B and AB are Fredholm operators, then again Y_0 and therefore Y_1 and Y_2 are finite dimensional. We also have that

$$\dim\ker(A) = \dim Y_2 + \dim(\operatorname{ran}(B)\cap\ker(A)) < \infty \tag{3.14}$$

by a) and $\operatorname{codim}\operatorname{ran}(A) = \dim Z_0 \leq \operatorname{codim}\operatorname{ran}(AB) < \infty$. So as before, the claim holds.
- Assuming that A and AB are Fredholm operators. Then by b) we know that $Z_0 \oplus AY_1$ is finite dimensional and hence Y_1 and Z_0 by themselves. Also Y_2 is finite dimensional which implies $Y_0 \simeq Y_1 \oplus Y_2$ is finite dimensional. So $\operatorname{codim}\operatorname{ran}(B)$ is finite and the claim holds yet again.

Before we continue, let us summarise that the calculations above show that both sequences

$$0 \longrightarrow \ker(B) \longrightarrow \ker(AB) \xrightarrow{B} \ker(A)\cap\operatorname{ran}(B) \longrightarrow 0, \tag{3.15}$$

and

$$0 \longrightarrow {}^{\operatorname{ran}(B)+\ker(A)}\!/_{\operatorname{ran}(B)} \longrightarrow {}^{Y}\!/_{\operatorname{ran}(B)} \xrightarrow{A} {}^{Z}\!/_{\operatorname{ran}(AB)} \longrightarrow {}^{Z}\!/_{\operatorname{ran}(A)} \longrightarrow 0 \tag{3.16}$$

are exact sequences. If two of the three operators A, B and AB are Fredholm operators, then at most one space in the above calculations is infinite dimensional. The exactness implies that this potential infinite dimensional space is finite dimensional.

d) We do a case study:

- Assume first that A and B are Fredholm operators. In this setting Y_3 is a closed space since $\operatorname{ran}(B)\cap\ker(A) \subset \ker(A)$ is finite dimensional. Since $A|_{Y_3\oplus Y_1}$ is bijective (onto $\operatorname{ran}(A)$) and $\operatorname{ran}(A)$ is closed by assumption, we know that Y_3 is a closed space. Since $\dim Y_1 \leq \dim Y_0 < \infty$, also Y_1 is a closed space and therefore $Y_3 \oplus Y_1$ is a Banach space. The restriction $A|_{Y_3\oplus Y_1}$ then becomes invertible (onto $\operatorname{ran}(A)$) and as the image of a closed space, the space AY_3 is itself closed. Since it holds $\operatorname{ran}(AB) = AY_3$, we get the closedness of $\operatorname{ran}(AB)$.
- Assume now that B and AB are Fredholm operators. Then again $\dim Y_1 \leq \dim Y_0 < \infty$. Therefore AY_1 is finite dimensional and hence closed. From the proof of b) we get $\operatorname{ran}(A) = \operatorname{ran}(AB) \oplus AY_1$, which is closed by assumption.

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- Assume now that A and AB are Fredholm operators. Let $X_0 \subset X$ be a closed space, such that $X = \ker(AB) \oplus X_0$. We take the inclusion map $J\colon X_0 \longrightarrow X$ and the restriction map $R\colon Z \longrightarrow \operatorname{ran}(AB)$ as in Proposition 2.6.

  We claim that $\operatorname{ran}(B) = \operatorname{ran}(BJ)$ is closed. Indeed let $y_k := BJx_k \in \operatorname{ran}(BJ)$ be a convergent sequence with $y_k \to y$ as $k \to \infty$. Then it follows $RAy_n \to RAy$ as $k \to \infty$. Note that RA is a Fredholm operator (follows from the first case study in this exercise combined with the previous results) and therefor $RAy \in \operatorname{ran}(RA)$. Define the map $C := RABJ\colon X_0 \longrightarrow \operatorname{ran}(AB)$. Then from previous results C is a Fredholm operator and bijective with continuous inverse. Hence

$$x_k = C^{-1}RAy_k \xrightarrow{k\to\infty} C^{-1}RAy =: x_* \in X_0. \tag{3.17}$$

  Therefore $y_k \to BJx_*$ and thus $\operatorname{ran}(BJ) = \operatorname{ran}(B)$ is closed.

e) If two of the three operators A, B and AB are Fredholm operators then we have shown that the ranges are in either case closed and it always holds

$$\dim\ker(AB) = \dim\ker(B) + \dim(\ker(A) \cap \operatorname{ran}(B)), \tag{3.18}$$
$$\operatorname{codim}\operatorname{ran}(AB) = \operatorname{codim}\operatorname{ran}(A) + \operatorname{codim}\operatorname{ran}(B) - \dim\ker(A) + \dim(\operatorname{ran}(B) \cap \ker(A)), \tag{3.19}$$

where all quantities are in either case finite. Therefore, all three operators must be Fredholm operators. With these identities we have

$$\begin{aligned}\operatorname{ind}(AB) &= \dim\ker(AB) - \operatorname{codim}\operatorname{ran}(AB) &(3.20)\\ &= \dim\ker(A) + \dim\ker(B) - \operatorname{codim}\operatorname{ran}(A) - \operatorname{codim}\operatorname{ran}(B) &(3.21)\\ &= \operatorname{ind}(A) + \operatorname{ind}(B). &(3.22)\end{aligned}$$

This concludes the proof. ■