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Functional Analysis 2 – Exercise Sheet 3

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Exercise 3.1

Let H be a Hilbert space and $A\colon \mathcal{D}(A) \longrightarrow H$ be a symmetric and closed operator. Show that only one of the following possibilities can occur:

$$i)\ \sigma(A) = \mathbb{C}, \qquad ii)\ \sigma(A) = \{\lambda \in \mathbb{C} : \Im(\lambda) \geq 0\},$$
$$iii)\ \sigma(A) \subset \mathbb{R}, \qquad iv)\ \sigma(A) = \{\lambda \in \mathbb{C} : \Im(\lambda) \leq 0\}.$$

Conclude that in this setting A is self-adjoint if and only if $\ker(A^* \pm \mathsf{i}\,\mathsf{id}) = \{0\}$.

Hint: Exercise 2.2 might be helpful.

Proof: Let $\mathbb{C}_\pm \coloneqq \{\lambda \in \mathbb{C} : \pm\Im(\lambda) > 0\}$ and let $\lambda \in \mathbb{C}_\pm$. Then we have by the identity $\|(A - \lambda\,\mathsf{id})x\|^2 = \|(A - \Re(\lambda)\,\mathsf{id})x\|^2 + \Im(\lambda)^2\,\|x\|^2$ that $\ker(A - \lambda\mathsf{id}) = \{0\}$, which implies that $A - \lambda\,\mathsf{id}$ is injective and using Exercise 2.2 this map also has a closed range.

Now, if $\operatorname{ran}(A - \lambda\,\mathsf{id}) = X$, then $\lambda \in \rho(A)$. But on the other hand, it also holds $\operatorname{ran}(A - \lambda\,\mathsf{id})^\perp = \ker(A^* - \bar{\lambda}\,\mathsf{id})$, which implies with the help of Exercise 2.2 that either $\mathbb{C}_\pm \subset \sigma(A)$, or $\mathbb{C}_\pm \cap \sigma(A) = \emptyset$.

Using the closedness of the spectrum (Theorem 1.10 (*ii*)) we conclude in the first case that only the three possibilities *i*), *ii*) and *iv*) can occur. In the latter case only possibility *iii*), which proves the characterisation of the spectra.

To prove the equivalence, let us first assume A is self-adjoint. Then by Proposition 1.16 (*iii*) we know $\sigma(A) \subset \mathbb{R}$ and therefore

$$\ker(A^* \pm \mathsf{i}\,\mathsf{id}) = \operatorname{ran}(A \mp \mathsf{i}\,\mathsf{id})^\perp = H^\perp = \{0\}. \tag{1.1}$$

So one directions holds. To prove the converse direction, assume $\ker(A^* \pm \mathsf{i}\,\mathsf{id}) = \{0\}$, which implies (since the range is closed)

$$\operatorname{ran}(A \pm \mathsf{i}\,\mathsf{id}) = \ker(A^* \mp \mathsf{i}\,\mathsf{id})^\perp = H. \tag{1.2}$$

So $A \pm \mathsf{i}\,\mathsf{id}$ is surjective. Using Proposition 1.16 (*ii*) then implies that A is self-adjoint. ■

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Exercise 3.2
Let X and Y be Banach spaces. Let $T \in \mathcal{L}(X,Y)$ and $T' \in \mathcal{L}(Y',X')$ be its adjoint.

a) Show that $\|T\|_{\mathcal{L}(X,Y)} = \|T'\|_{\mathcal{L}(Y',X')}$.

b) Show that T is a compact operator if and only if T' is a compact operator.

Hint: Statement b) is the statement of Proposition 1.26 (*iii*). You are allowed to use the remaining statements of that proposition for this exercise.

Proof:

a) Let $x \in X$ and $y' \in Y'$, then it holds

$$|\langle T'y', x\rangle| = |\langle y', Tx\rangle| \leq \|T\|_{\mathcal{L}(X,Y)} \, \|y'\|_{Y'} \, \|x\|_X. \tag{2.1}$$

Therefore $\|T'y'\|_{X'} \leq \|T\|_{(X,Y)}\|y'\|$ and thus $\|T\|_{\mathcal{L}(X,Y)} \geq \|T'\|_{\mathcal{L}(Y',X')}$.

Similarly we have

$$|\langle y', Tx\rangle| = |\langle T'y', x\rangle| \leq \|T'\|_{\mathcal{L}(Y',X')} \, \|y'\|_{Y'} \, \|x\|_X. \tag{2.2}$$

If $Tx = 0$, then we have nothing to show. If $Tx \neq 0$, then by the Hahn–Banach theorem there exists $z \in Y'$ such that $\|z\|_{Y'} = 1$ and $\langle z, Tx\rangle = \|Tx\|_Y$. Substituting $y' = z$ in (2.2) results in

$$\|Tx\|_Y \leq \|T'\|_{\mathcal{L}(Y',X')} \, \|x\|_X, \tag{2.3}$$

and thus $\|T\|_{\mathcal{L}(X,Y)} \leq \|T'\|_{\mathcal{L}(Y',X')}$. Altogether the claim follows.

b) Let T be a compact operator. Then by Proposition 1.26 (*ii*) we can find a sequence of compact operators of finite rank $(T_n)_n$ such that $T_n \to T$ in $\mathcal{L}(X,Y)$. Then the adjoints T_n' are also compact operators of finite rank for all $n \in \mathbb{N}$ and we have with the help of a) that $\|T' - T_n'\|_{\mathcal{L}(Y',X')} = \|T - T_n\|_{\mathcal{L}(X,Y)}$. So we can approximate T' by compact operators of finite rank and thus T' is a compact operator.

Conversely, assume T' is a compact operator. As previously shown, it follows that $(T')'$ is a compact operator. But $(T')'$ is an extension of T, so T is a compact operator. ■

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Exercise 3.3
Let H be a Hilbert space and let $A \in \mathcal{L}(H)$ be self-adjoint. Show that

$$\|A\| = \sup_{\lambda\in\sigma(A)} |\lambda| = \sup_{\|x\|=1} |(x, Ax)|. \tag{3.1}$$

Hint: Exercise 1.4 might be helpful. Use that $4\Re(x, Ay) = (x + y, Ax + Ay) - (x - y, Ax - Ay)$ and the parallelogram law to deduce

$$|(x, Ay)| \leq \sup_{\|z\|=1} |(z, Az)|$$

for $x, y \in H$ such that $\|x\| = \|y\| = 1$.

Proof: For self-adjoint operators it holds[1] $\|A^2\| = \|A\|^2$ and therefore $\|A^{2^n}\| = \|A\|^{2^n}$ for all $n \in \mathbb{N}$. From Exercise 1.4 we know that

$$\sup_{\lambda\in\sigma(A)} |\lambda| = \lim_{m\to\infty} \|A^m\|^{\frac{1}{m}} = \lim_{n\to\infty} \|A^{2^n}\|^{2^{-n}} = \|A\|. \tag{3.2}$$

Moreover it holds

$$|(x, Ax)| \leq \|A\| \, \|x\|^2 \quad \Longrightarrow \quad \sup_{\|x\|=1} |(x, Ax)| \leq \|A\|. \tag{3.3}$$

So it remains to show the reverse inequality. For this purpose, let $\alpha > 0$ and $x \in H$ with $x \notin \ker(A)$. Then we can compute

$$(A(\alpha x \pm \tfrac{1}{\alpha}Ax), \alpha x \pm \tfrac{1}{\alpha}Ax) = (A(\alpha x), \alpha x) + (A^2(\tfrac{1}{\alpha}x), A(\tfrac{1}{\alpha}x)) \pm 2\,\|Ax\|^2. \tag{3.4}$$

Taking the difference of those two equations results in the polarisation identity

$$4\,\|Ax\|^2 = (A(\alpha x + \tfrac{1}{\alpha}Ax), \alpha x + \tfrac{1}{\alpha}Ax) - (A(\alpha x - \tfrac{1}{\alpha}Ax), \alpha x - \tfrac{1}{\alpha}Ax). \tag{3.5}$$

For simplicity, define the following quantities:

$$s := \sup_{\|x\|=1} |(x, Ax)|, \qquad u_\pm := \alpha x \pm \tfrac{1}{\alpha}Ax, \qquad \tilde{u}_\pm := \frac{u_\pm}{\|u_\pm\|}. \tag{3.6}$$

Note that $u_\pm \neq 0$, from which we conclude that $\tilde{u}_\pm$ is well defined with $\|\tilde{u}_\pm\| = 1$. We obtain from (3.5) together with the parallelogramm law

$$\begin{aligned} 4\,\|Ax\|^2 &= \|u_+\|^2\,(A\tilde{u}_+, \tilde{u}_+) - \|u_-\|^2\,(A\tilde{u}_-, \tilde{u}_-) && (3.7)\\ &\leq s\,(\|u_+\|^2 + \|u_-\|^2) && (3.8)\\ &= 2s\,(\alpha^2\|x\|^2 + \tfrac{1}{\alpha^2}\|Ax\|^2). && (3.9) \end{aligned}$$

Now choosing $\alpha = \frac{\|Ax\|}{\|x\|}$ results in

$$4\,\|Ax\|^2 \leq 4s\,\|Ax\|\,\|x\|, \tag{3.10}$$

from which the claim follows. ■

[1] Follows from $\|A^2\| = \sup_{\|x\|=1} (A^2x, x) = \sup_{\|x\|=1} \|Ax\|^2 = \|A\|^2$.

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Exercise 3.4
For a Hilbert space H and a non-negative operator $A \in \mathcal{L}(H)$ recall the notation $|A| := (A^*A)^{\frac{1}{2}}$. Let $H = \mathbb{C}^2$ and define the matrices $X, Y \in \mathcal{L}(H)$ via

$$X := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Y := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.1}$$

a) Compute $|X + \mathsf{id}|$ and $|Y - \mathsf{id}|$.

b) Show that it does not hold $|(X + \mathsf{id}) + (Y - \mathsf{id})| \leq |X + \mathsf{id}| + |Y - \mathsf{id}|$.

Proof:

a) Since X, Y and id are symmetric and have entries in $\mathbb{R}$, it holds $(X + \mathsf{id})^* = X + \mathsf{id}$ and $(Y - \mathsf{id})^* = Y - \mathsf{id}$. We compute

$$(X + \mathsf{id})^2 = 4\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad (Y - \mathsf{id})^2 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}. \tag{4.2}$$

So we get the desired quantities by taking the operator square root. Notice that we want the square root to be ≥ 0. We get

$$|X + \mathsf{id}| = 2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad |Y - \mathsf{id}| = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{4.3}$$

b) We need to compute

$$|X + Y| = \sqrt{\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}} = \sqrt{2}\,\mathsf{id}. \tag{4.4}$$

Now it holds

$$|X + \mathsf{id}| + |Y - \mathsf{id}| - |X + Y| = \begin{pmatrix} 3 - \sqrt{2} & -1 \\ -1 & 1 - \sqrt{2} \end{pmatrix}. \tag{4.5}$$

We compute the eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -2(\sqrt{2} - 1)$. Therefore the matrix is indefinite and thus it does not hold $|(X + \mathsf{id}) + (Y - \mathsf{id})| \leq |X + \mathsf{id}| + |Y - \mathsf{id}|$. ■