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Functional Analysis 2 – Exercise Sheet 2

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Exercise 2.1
Let H be a separable Hilbert space and let $A\colon \mathcal{D}(A) \longrightarrow H$ and $B\colon \mathcal{D}(B) \longrightarrow H$ be two symmetric operators such that $A \subset B$. Show that $B^* \subset A^*$ and conclude that any symmetric extension of a self-adjoint operator is the operator itself.

Proof: We know that $B \subset B^*$, and therefore

$$\mathcal{D}(A) \subset \mathcal{D}(B) \subset \mathcal{D}(B^*) = \left\{ y \in H \colon \exists \tilde{y} \in H : (Bx, y) = (x, \tilde{y}) \ \forall x \in \mathcal{D}(B) \right\}. \tag{1.1}$$

Let $y \in \mathcal{D}(B^*)$, then we have $(Bx, y) = (x, B^*y)$ for all $x \in \mathcal{D}(B)$, and therefore for all $x \in \mathcal{D}(A)$. Hence there exists $z = B^*y \in H$ such that

$$(Ax, y) = (Bx, y) = (x, z) \qquad \text{for all } x \in \mathcal{D}(A). \tag{1.2}$$

This implies $y \in \mathcal{D}(A^*)$, so $\mathcal{D}(B^*) \subset \mathcal{D}(A^*)$. The preceeding argument also shows $B^* = A^*$ in $\mathcal{D}(B^*)$, so that $B^* \subset A^*$. Altogether we have the following chains:

$$\mathcal{D}(A) \subset \mathcal{D}(B) \subset \mathcal{D}(B^*) \subset \mathcal{D}(A^*), \qquad A \subset B \subset B^* \subset A^*. \tag{1.3}$$

Now, if A is self-adjoint, then we have $\mathcal{D}(A) = \mathcal{D}(A^*)$ and so all "$\subset$" in (1.3) become "$=$" which proves the claim. ■

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Exercise 2.2
Let H be a Hilbert space and $A\colon \mathcal{D}(A) \longrightarrow H$ be a symmetric and closed operator. In this exercise we want to show that the function $\lambda \longmapsto \dim\ker(A^* - \lambda\,\mathsf{id})$ is constant on the half spaces $\mathbb{C}_\pm := \{\lambda \in \mathbb{C} : \pm\Im(\lambda) > 0\}$. We will restrict ourselves to the upper halfspace $\mathbb{C}_+$, the argument for $\mathbb{C}_-$ is analogueous.

a) Let $M, N \subset H$ be closed subspaces such that $M \cap N^\perp = \{0\}$. Show that $\dim M \leq \dim N$.

b) Let $\lambda \in \mathbb{C}$ such that $\Im(\lambda) \neq 0$. Show that $\operatorname{ran}(A - \lambda\,\mathsf{id})$ is closed.

c) Let $\lambda \in \mathbb{C}_+$ and $\mu \in \mathbb{C}$ such that $|\lambda - \mu| < \Im(\lambda)$. Show that $\ker(A^* - \mu\,\mathsf{id}) \cap \ker(A^* - \lambda\,\mathsf{id})^\perp = \{0\}$.

d) Show that $\lambda \longmapsto \dim(A^* - \lambda\,\mathsf{id})$ is locally constant on $\mathbb{C}_+$ and conclude that this map is actually constant on the whole halfspace $\mathbb{C}_+$.

Hint: Recall the identity $\|(A - \lambda\,\mathsf{id})x\|^2 = \|(A - \Re(\lambda)\,\mathsf{id})x\|^2 + \Im(\lambda)^2\,\|x\|^2$. You can use that it holds $\operatorname{ran}(A^*)^\perp = \ker(A)$ for closed operators. In order to show c) use an indirect proof.

Proof: Let us first recall that for $\lambda = \alpha + \mathbf{i}\beta$ and $x \in \mathcal{D}(A)$ we have

$$\|(A - \lambda\,\mathsf{id})x\|^2 = \|(A - \alpha\,\mathsf{id})x - \mathbf{i}\beta\,x\|^2 = \|(A - \alpha\,\mathsf{id})x\|^2 + \beta^2\|x\|^2 + 2\Re[\mathbf{i}\,((A - \alpha\,\mathsf{id})x, \beta x)]. \tag{2.1}$$

Since A is symmetric we have $(x, Ax) \in \mathbb{R}$ for all $x \in H$ and therefore

$$((A - \alpha\,\mathsf{id})x, \beta x) = \beta(Ax, x) - \alpha\beta\|x\|^2 \in \mathbb{R}, \tag{2.2}$$

which means $\Re[\mathbf{i}\,((A - \alpha\,\mathsf{id})x, \beta x)] = 0$ and it holds

$$\|(A - \lambda\,\mathsf{id})x\|^2 = \|(A - \alpha\,\mathsf{id})x\|^2 + \beta^2\|x\|^2 \qquad \text{for all } x \in H. \tag{2.3}$$

a) Let $\mathbb{P}\colon H \longrightarrow N$ be the orthogonal projection and define the map $T\colon M \longrightarrow N$ via $Tx := \mathbb{P}x$ for all $x \in M$. We want to show that T is injective and we choose $x, y \in M$ such that $x \neq y$, or $x - y \neq 0$. By assumption $x - y \notin N^\perp$ and therefore $T(x - y) \neq 0$, so that T is injective. Now given a linear subspace of $L \subset M$ we have $\dim L = \dim TL \leq \dim N$, so that $\dim M \leq \dim N$ (in case N is infinite dimensional, otherwise the statement is clear).

b) Using (2.3) we see $\|(A-\lambda\,\mathsf{id})x\|^2 \geq \Im(\lambda)^2\,\|x\|^2$. Now let $(x_n)_n \subset \mathcal{D}(A)$ such that $(A-\lambda\,\mathsf{id})x_n \to y$. By the preceeding inequality we obtain that $(x_n)_n$ is a Cauchy-sequence and is therefore convergent with say limit $x \in H$. Since A is closed we obtain that $y = (A - \lambda\,\mathsf{id})x \in \operatorname{ran}(A - \lambda\,\mathsf{id})$ which shows the closedness.

c) Let $\beta := \Im(\lambda)$. We assume the contrary: let $0 \neq x \in \ker(A^* - \mu\,\mathsf{id}) \cap \ker(A^* - \lambda\,\mathsf{id})^\perp$. We can renomalise x such that $\|x\| = 1$. Using b) and the hint we know that $\ker(A^*-\lambda\,\mathsf{id})^\perp = \operatorname{ran}(A-\bar\lambda\,\mathsf{id})$, so we can find $y \in \mathcal{D}(A)$ such that $x = (A - \bar\lambda\,\mathsf{id})y$. Since $x \in \ker(A^* - \mu\,\mathsf{id})$ we have

$$
\begin{aligned}
0 &= ((A^* - \mu\,\mathsf{id})x, y) = (x, (A - \bar\mu\,\mathsf{id})y) && (2.4)\\
&= (x, (A - \bar\lambda + \bar\lambda - \bar\mu\,\mathsf{id})y) && (2.5)\\
&= \|x\|^2 + (\lambda - \mu)\,(x, y) + (x, (A - \bar\lambda\,\mathsf{id})y) && (2.6)\\
&= \|x\|^2 + (\lambda - \mu)\,(x, y) && (2.7)
\end{aligned}
$$

since $x \in \ker(A^* - \lambda\,\mathsf{id})^\perp$ by assumption.
Now, on the one hand we have using the Cauchy-Schwarz inequality

$$1 = \|x\|^2 \leq |\mu - \lambda|\,|(x, y)| \leq |\mu - \lambda|\,\|x\|\,\|y\| < \beta\,\|y\| \quad \Longrightarrow \quad \|y\| > \frac{1}{\beta}. \tag{2.8}$$

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On the other hand we obtain by (2.3)

$$1 = \|x\| = \|(A - \bar{\lambda}\,\mathsf{id})y\| \geq \beta\,\|y\| \quad \Longrightarrow \quad \|y\| \leq \frac{1}{\beta}. \tag{2.9}$$

So (2.8) and (2.9) are contradicting each other and therefore the claim holds.

d) Using c) we obtain $\dim\ker(A^* - \mu\,\mathsf{id}) \leq \dim\ker(A^* - \lambda\,\mathsf{id})$ provided $|\lambda - \mu| < \Im(\lambda)$. Interchanging the role of λ and μ shows that $\dim\ker(A^* - \mu\,\mathsf{id}) = \dim\ker(A^* - \lambda\,\mathsf{id})$ locally. Since $\mathbb{C}_+$ is a connected subset of $\mathbb{C}$ we get the claim. ■

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Exercise 2.3
Let $\mathcal{H} := L^2((0,1),\mathbb{C})$, $\mathcal{D}(A) := H_0^2((0,1),\mathbb{C})$ and $\mathcal{D}(B) := H_0^1((0,1),\mathbb{C})$. Let $\varepsilon > 0$ and define the operators $A\colon \mathcal{D}(A) \longrightarrow \mathcal{H}$ and $B\colon \mathcal{D}(B) \longrightarrow \mathcal{H}$ via

$$A := -\Big(\frac{\mathrm{d}}{\mathrm{d}x}\Big)^2, \qquad\qquad B := \mathbf{i}\varepsilon\,\frac{\mathrm{d}}{\mathrm{d}x}. \tag{3.1}$$

Show that $A + B\colon \mathcal{D}(A) \longrightarrow \mathcal{H}$ is a self-adjoint operator.

Proof: From Example 1.21 we know that A is a self-adjoint operator. Using integration by parts we also see that for $f, g \in \mathcal{D}(B)$

$$(Bf, g) = \mathbf{i}\varepsilon \int_0^1 f'(x)\,\bar{g}(x)\,\mathrm{d}x = [f(1)\,\bar{g}(1) - f(0)\,\bar{g}(0)] - \mathbf{i}\varepsilon \int_0^1 f(x)\,\bar{g}'(x)\,\mathrm{d}x = (f, Bg). \tag{3.2}$$

So B is a symmetric operator. We want to use the theorem of Kato-Rellich, so it remains to show that B is A – continuous. For this purpose, let $f \in \mathcal{D}(A)$. We compute

$$\|Bf\|^2 = \varepsilon^2 \int_0^1 |f'(x)|^2\,\mathrm{d}x = \varepsilon^2 \int_0^1 f'(x)\,\bar{f}'(x)\,\mathrm{d}x \tag{3.3}$$

$$= \varepsilon^2[f'(1)\,\bar{f}(1) - f'(0)\,\bar{f}(0)] - \varepsilon^2 \int_0^1 f''(x)\,\bar{f}(x)\,\mathrm{d}x \tag{3.4}$$

$$= \varepsilon^2(Af, f) \le \varepsilon^2\|Af\|\,\|f\|. \tag{3.5}$$

Now let $0 < \delta < \frac{\varepsilon^2}{2}$. Then we can use Young's inequality (with $p = q = 2$) to obtain

$$\varepsilon^2\|Af\|\,\|f\| = \delta\varepsilon^2\|Af\|\,\frac{1}{\delta}\|f\| \le \frac{1}{2}\Big(\delta^2\,\varepsilon^4\,\|Af\|^2 + \frac{1}{\delta^2}\,\|f\|^2\Big) < \frac{1}{4}\|Af\|^2 + C_\delta^2\|f\|^2, \tag{3.6}$$

where $C_\delta^2 := (2\delta^2)^{-1}$. Taking the square root we obtain

$$\|Bf\| \le \frac{1}{2}\|Af\| + C_\delta\,\|f\| \qquad\qquad \text{for all } f \in \mathcal{D}(A). \tag{3.7}$$

Using the theorem of Kato-Rellich proves the claim. ■

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Exercise 2.4
Let $H = L^2(\mathbb{R}^n, \mathbb{C})$ and $A \coloneqq \mathcal{D}(A) \longrightarrow H$ be a self-adjoint operator. Let $\psi \in C^1([0, \infty), \mathcal{D}(A))$ be a solution to the time-dependent homogeneous Schrödinger equation

$$\mathbf{i} \frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = A\psi(t) \qquad \text{for all } t > 0 \tag{4.1}$$

with initial data $\psi(0) = \psi_0 \in \mathcal{D}(A)$. Let $\|\psi_0\| = 1$. Show that $\|\psi(t)\| = 1$ for all $t > 0$.

Proof: We see that

$$-\mathbf{i}\frac{\mathrm{d}}{\mathrm{d}t}\|\psi(t)\|^2 = (\psi(t), A\psi(t)) = (A\psi(t), \psi(t)) = \mathbf{i}\frac{\mathrm{d}}{\mathrm{d}t}\|\psi(t)\|^2, \tag{4.2}$$

so that $\frac{\mathrm{d}}{\mathrm{d}t}\|\psi(t)\|^2 = 0$, which means $t \longmapsto \|\psi(t)\|$ is constant. Since $\|\psi_0\| = 1$ we conclude that $\|\psi(t)\| = 1$ for almost all $t > 0$. ■