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# Functional Analysis 2 – Exercise Sheet 1

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**Exercise 1.1**
Let $H$ be a Hilbert space and $(x_n)_n \subset H$ a weakly convergent sequence with limit $x \in H$. Show the following statements:

a) $\|x\| \leq \liminf_{n\to\infty} \|x_n\|$.

b) If $\|x_n\| \to \|x\|$, then it holds $x_n \to x$ in $H$ strongly.

c) Let $(y_n)_n \subset H$ be a strongly convergent sequence in $H$ with limit $y$. Show that $(x_n, y_n) \to (x, y)$.

**Proof:**

a) Since $(x_n)_n$ is weakly convergent it is also a bounded sequence so that we can extract a subsequence $(x_{n_j})_j$ such that

$$\lim_{j\to\infty} \|x_{n_j}\| = \liminf_{n\to\infty} \|x_n\|. \tag{1.1}$$

Using the inequality of Cauchy–Schwarz results in

$$\|x\|^2 = (x, x) = \lim_{n\to\infty} (x, x_n) = \lim_{j\to\infty} (x, x_{n_j}) \leq \|x\| \lim_{j\to\infty} \|x_{n_j}\| = \|x\| \liminf_{n\to\infty} \|x_n\|. \tag{1.2}$$

b) We can compute

$$\|x - x_n\|^2 = (x - x_n, x - x_n) = (x, x) - (x, x_n) - (x_n, x) + (x_n, x_n) \tag{1.3}$$

$$= \|x\|^2 + \|x_n\|^2 - (x_n, x) - (x, x_n) \xrightarrow{n\to\infty} 0. \tag{1.4}$$

The second term in (1.4) converges by assumption and the remaining one by definition of weak convergence.

c) We can compute

$$|(x_n, y_n) - (x, y)| = |(x_n, y_n) - (x_n, y) + (x_n, y) - (x, y)| \leq |(x_n, y_n - y)| + |(x_n - x, y)|. \tag{1.5}$$

The second term on the right hand side converges to 0 by definition of weak convegerce. The first term also converges to 0. Indeed, using the inequality of Cauchy–Schwarz, the assumption on the convergence and the boundednes of weakly convergent sequences we get

$$|(x_n, y_n - y)| \leq \|x_n\| \, \|y_n - y\| \lesssim \|y_n - y\| \xrightarrow{n\to\infty} 0, \tag{1.6}$$

from which the claim follows. ∎

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**Exercise 1.2**
Let $H$ be a seperable Hilbert space and $\{w_n\}_n \subset H$ an orthonormal system of pairwise distinct elements. Let $A \colon H \longrightarrow H$ be a compact operator. Show that $Aw_n \to 0$ in $H$ strongly.

**Proof:** Since $H$ is separable we can expand $\{w_n\}_n$ to a Hilberstpace basis $\{v_\ell\}_\ell$ using the method of Gram–Schmidt. Also, since $H$ is reflexive, we know that weak and weak* convergence are the same. Now, let $x \in H$ be arbitrary, then $x$ can be represented via

$$x = \sum_{\ell \in \mathbb{N}} c_\ell \, v_\ell \tag{2.1}$$

for constants $c_\ell \in \mathbb{K}$. Necessary for convergence, the sequecen $(c_\ell)_\ell$ converges to 0. So it follows

$$\langle v_k, x \rangle = \sum_{\ell \in \mathbb{N}} c_\ell \langle v_k, v_\ell \rangle = c_k \xrightarrow{k \to \infty} 0. \tag{2.2}$$

Since $x$ was arbitrary, it follows that $\{v_\ell\}_\ell$ is weakly convergent to 0 and therefore also $\{w_n\}_n$ as subsequence. Since compact operators map weakly convergent sequences to strongly convergent sequences, the claim follows.[1] ■

[1] C.f. [Alt, Lemma 10.2].

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**Exercise 1.3**

Let $X$ be a Banach space and $A \in \mathcal{L}(X)$. In this exercise we want to show that

$$\lim_{m\to\infty} \|A^m\|_{X'}^{\frac{1}{m}} = \inf_{m\in\mathbb{N}} \|A^m\|_{X'}^{\frac{1}{m}}, \tag{3.1}$$

in particular, that the limit exists. For this purpose we define the sequence

$$a_m := \log \|A^m\|_{X'} \qquad \text{for all } m \in \mathbb{N}. \tag{3.2}$$

a) Show that $a_{n+m} \leq a_n + a_m$ for all $m, n \in \mathbb{N}$.

b) Let $n = mq + r$, where $n, m, q, r \in \mathbb{N}$ and $0 \leq r \leq m - 1$. Show that

$$\limsup_{n\to\infty} \frac{a_n}{n} \leq \frac{a_m}{m}. \tag{3.3}$$

c) Show that

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\in\mathbb{N}} \frac{a_n}{n} \tag{3.4}$$

and conclude (3.2).

<u>**Proof:**</u> We denote $\|\ \cdot\ \| := \|\ \cdot\ \|_{X'}$.

a) Since the operator norm is submultiplicative, we get for $n, m \in \mathbb{N}$

$$a_{n+m} = \log \|A^{m+n}\| \leq \log(\|A^n\|\,\|A^m\|) = \log(\|A^n\|) + \log(\|A^m\|) = a_n + a_m. \tag{3.5}$$

b) Let $n = qm + r$, then we have

$$\frac{a_n}{n} = \frac{a_{qm+r}}{n} \overset{\text{a)}}{\leq} \frac{a_{qm}}{n} + \frac{a_r}{n}. \tag{3.6}$$

From the subadditivity of the operator norm and the monotonicity of the logarithm we get

$$a_{qm} = \log \|A^{qm}\| \leq q \log \|A^m\| = q\, a_m, \qquad a_r \leq r\, a_1, \tag{3.7}$$

from which follows

$$(3.6) \leq \frac{a_m}{m} \frac{q}{q + \frac{r}{m}} + a_1 \frac{r}{n}. \tag{3.8}$$

For fixed $m$, the parameter $r$ is bounded so that for $q \to \infty$ we have for the second term $a_1 \frac{r}{n} \to 0$. The factor $\frac{q}{q+\frac{r}{m}}$ converges to 1 as $q \to \infty$. Altogether, the claim follows.

c) Since every number $n \in \mathbb{N}$ has a representation as in b), we see that

$$\inf_{n\in\mathbb{N}} \frac{a_n}{n} \leq \frac{a_n}{n} \leq \limsup_{q\to\infty} \frac{a_n}{n} \leq \min_{m\in\mathbb{N}} \frac{a_m}{m}. \tag{3.9}$$

Using the sandwich lemma implies the claim. ■

**Exercise 1.4**
Let $X$ be a Banach space and $A \in \mathcal{L}(X)$. Show that

$$\sup_{\lambda \in \sigma(A)} |\lambda| = \lim_{m \to \infty} \|A^m\|_{X'}^{\frac{1}{m}}. \tag{4.1}$$

*Hint:* Compute the radius of convergence of the Laurent series of the resolvent with the formula of Cauchy-Hadamard for Banach space valued functions and argue why the radius of convergence is given by the spectral radius. Use the analyticity of the resolvent.

**Proof:** We denote $r(A)$ for the left hand side and write $\| \cdot \| := \| \cdot \|_{X'}$.

Using Theorem 1.10, the sepcturm $\sigma(A)$ is compact, and $r(A)$ is therefore finite. Let $\lambda \in \mathbb{C}$ and $R_\lambda$ be the resolvent of $A$ with respect to $\lambda$. Using the Neumann series we get for $|\lambda| > \|A\|$

$$R_\lambda = -\frac{1}{\lambda}\Big[\mathsf{id} + \sum_{k=1}^{\infty} \Big(\frac{A}{\lambda}\Big)^k\Big]. \tag{4.2}$$

This is a Laurent series. Our goal is to determine the inner radius of convergence and to show that this is equal to $r(A)$ as well as the right hand side of (4.1).

If $|\lambda > r(A)$, then $R_\lambda$ is analytic and using Theorem 1.10, we know $\lambda \in \rho(A)$. This implies, that the series (4.2) converges, the inner radius of convergence must therefore be greater or equal than $r(A)$.

On the other hand, $R_\lambda$ exists if (4.2) converges. Is $\lambda \in \sigma(A)$, then (4.2) does not converge since the resolvent does not exist. We can therefore find a sequence $(\lambda_j)_j \subset \sigma(A)$ with $|\lambda_j| \nearrow r(A)$ for $j \to \infty$ and $R_{\lambda_j}$ does not exist. So the series in (4.2) does not exist, from which we conclude, that the inner radius of convergence must be $r(A)$.

To show the identity (4.1), we use the Cauchy–Hadamrd formula

$$r = \limsup_{k \to \infty} \|A^k\|^{\frac{1}{k}} = \lim_{k \to \infty} \|A^k\|^{\frac{1}{k}}, \tag{4.3}$$

where in the last equality we used Exercise 1.3. This proves $r = r(A)$. ∎