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Exercise	8.1	8.2	8.3	8.4	$\sum$
Points:					

Functional Analysis 2 – Exercise Sheet 8

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Exercise 8.1

Let $P\colon \mathbb{R} \longrightarrow \mathbb{R}$ be the Poisson kernel, given by $P(x) := \frac{1}{\pi}\frac{1}{1+x^2}$ for all $x \in \mathbb{R}$. For $t > 0$ let $P_t(x) := \frac{1}{t}\, P(\frac{x}{t})$ be its L^1-dilation.

a) Show that $\hat{P}(\xi) = \frac{1}{\sqrt{2\pi}} e^{-|\xi|}$ for all $\xi \in \mathbb{R}$.

b) Let $f \in L^2(\mathbb{R})$. Show that the function $u(x,t) := (P_t * f)(x)$ solves the problem

$$\begin{cases} (\partial_t^2 + \partial_x^2)\, u(x,t) = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = f(x) & \text{for almost all } x \in \mathbb{R}. \end{cases} \tag{1.1}$$

c) Let $T\colon [0,\infty) \longrightarrow L^2(\mathbb{R})$ be given by $T(0) = \mathsf{id}$ and $T(t)f := P_t * f$ for all $t > 0$. Show that T is a C^0–semigroup on $L^2(\mathbb{R})$.

Exercise 8.2

Let X be a Banach space and let T be a C^0–semigroup. For every $t > 0$ let $T(t)$ be invertible and $T(t)^{-1} \in \mathcal{L}(X)$.

a) Show that $S\colon [0,\infty) \longrightarrow \mathcal{L}(X)$ defined by $S(t) := T(t)^{-1}$ is a C^0-semigroup.

b) Let A be the generator of T. Show that $-A$ is the generator of S.

c) Define $U\colon \mathbb{R} \longrightarrow \mathcal{L}(X)$ via

$$U(t) := \begin{cases} T(t) & \text{if } t \geq 0, \\ S(-t) & \text{if } t < 0. \end{cases} \tag{2.1}$$

Show that U is a C^0–group.

Exercise 8.3

Let $T\colon [0,\infty) \longrightarrow \mathcal{L}(L^2(\mathbb{R}))$ be defined by $T(t)f := f(t + \,\cdot\,)$ for all $t \geq 0$.

a) Show that T is a C^0–semigroup of contractions on $L^2(\mathbb{R})$.

b) Show that the operator $Af := f'$ on the domain $\mathcal{D}(A) := H^1(\mathbb{R})$ is the generator of T.

c) Show the inequality of Landau–Kolmogorov: There exists a constant $C > 0$ such that for every $f \in H^2(\mathbb{R})$ it holds

$$\|f'\|_{L^2(\mathbb{R})}^2 \leq 4\, \|f\|_{L^2(\mathbb{R})}\, \|f''\|_{L^2(\mathbb{R})}. \tag{3.1}$$

Exercise 8.4

Let X be a Banach space. Let T be a C^0–semigroup and let $A\colon \mathcal{D}(A) \longrightarrow X$ be its generator. Show Taylor's formula

$$T(t)x = x + tAx + \int_0^t (t-s)\, T(s) A^2 x \,\mathrm{d}s \qquad \text{for all } x \in \mathcal{D}(A^2). \tag{4.1}$$