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Exercise	7.1	7.2	7.3	$\sum$
Points:				

Functional Analysis 2 – Exercise Sheet 7

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Exercise 7.1

Let $H = L^2(\mathbb{R}^n, \mathbb{C})$ and $\mathcal{F} \colon H \longrightarrow H$ be the Fourier transform. Show that $\sigma_p(\mathcal{F}) = \{\pm 1, \pm \mathbf{i}\}$.

Hint: Apply the Fourier transform several times to the eigenvalue equation. Compute the Fourier transforms of functions of the form $x\, e^{-x^2}$, $(1+x^2)\, e^{-x^2}$ and $x\,(1+x^2)\, e^{-x^2}$.

Exercise 7.2

Let $j \in \{1, \dots, n\}$ and let $f \in L^2(\mathbb{R}^n, \mathbb{C})$. Show that there exists a unique solution $u \in H^1(\mathbb{R}^n, \mathbb{C})$ such that $(\mathbf{i} \pm \mathbf{i}\partial_j)u = f$ almost everywhere in $\mathbb{R}^n$.

Exercise 7.3

Let X be a Banach space and $T \in \mathcal{L}(X)$. Let $f \colon \mathbb{R}^n \longrightarrow X$ be Bochner integrable. Show that Tf is a Bochner integrable function and that

$$\int_{\mathbb{R}^n} Tf(x)\,\mathrm{d}x = T\Big(\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x\Big). \tag{3.1}$$

Hint: A quick repetition of the Bochner integration theory: A function $g \colon \mathbb{R}^n \longrightarrow X$ is called *simple* if there exist finitely many Lebesgue measurable sets $\{A_j\}_j$ with $|A_j| < \infty$ and elements $\{\alpha_j\}_j \subset X$, such that $g = \sum_j \alpha_j\, \chi_{A_j}$. For simple functions we define the Bochner integral

$$\int_{\mathbb{R}^n} g(x)\,\mathrm{d}x := \sum\nolimits_j \alpha_j\, |A_j|. \tag{3.2}$$

A function $h \colon \mathbb{R}^n \longrightarrow X$ is called *Bochner measurable* if there exists a sequence $(g_k)_k$ of simple functions, such that $g_k \to h$ pointwise almost everywhere. A Bochner measurable function $f \colon \mathbb{R}^n \longrightarrow X$ is called *Bochner integrable* if there exists a sequence of simple functions $(g_k)_k$, such that $g_k \to f$ almost everywhere and

$$\int_{\mathbb{R}^n} \|g_i(x) - g_j(x)\|\,\mathrm{d}x \to 0 \qquad \text{as } i, j \to \infty. \tag{3.3}$$

We then define the Bochner integral of f via

$$\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x := \lim_{k\to\infty} \int_{\mathbb{R}^n} g_k(x)\,\mathrm{d}x. \tag{3.4}$$

In order to show the statement: consider first simple functions and then go to the limit.