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Exercise	6.1	6.2	6.3	$\sum$
Points:				

Functional Analysis 2 – Exercise Sheet 6

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Exercise 6.1

Let H be a separable Hilbertspace and $A \in \mathcal{L}(H)$ a self–adjoint operator. Let $\{\mu_n\}_n$ be the (possibly inifinite) collection of spectral measures with respect to A. Show that

$$\sigma(A) = \overline{\bigcup_n \operatorname{spt} \mu_n}. \tag{1.1}$$

Exercise 6.2

Let $H := L^2((0,1), \mathbb{C})$ and let $h \in H$. Let $\lambda \in \mathbb{C}$. In this exercise we want to solve the Sturm–Liouville problem

$$\begin{cases} u''(x) + \lambda u(x) = h(x) & \text{for almost all } x \in (0,1), \quad \text{(2.1a)} \\ u(0) = u(1) = 0. & \text{(2.1b)} \end{cases}$$

Using methods of the theory of ordinary differential equations we know that the problem (2.1) is equivalent to the integral equation $u - \lambda Ku = -Kh$, where $K \colon H \longrightarrow H$ is an integral operator defined by

$$Kf(x) := \int_0^1 k(x,t)\, f(t)\, \mathrm{d}t, \qquad k(x,t) := \min\{x,t\} - xt. \tag{2.2}$$

a) Determine the spectrum of K.

b) Give necessary and sufficient conditions on λ so that (2.1) has a unique solution.

c) Give a representation formula for the solution to (2.1) in terms of h and λ.

Exercise 6.3

Let X be a Banach space, $\mathcal{D}(A) \subset X$ a dense subspace and $A \colon \mathcal{D}(A) \longrightarrow X$ a linear operator. Let $\rho(A) \neq \emptyset$. Show that A is a closed operator.