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Exercise	5.1	5.2	5.3	5.4	5.5	5.6	$\sum$
Points:							

Functional Analysis 2 – Exercise Sheet 5

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Exercise 5.1

Let H be a separable Hilbert space and $A \in \mathcal{J}_2(H)$. Show that A is a compact operator.

Hint: Find an approximation of finite–rank operators.

Exercise 5.2

Let X be a Banach space and let $K \in \mathcal{L}(X)$ be a compact operator. Let $U \subset X$ be open, such that $0 \in U$ and let $N \colon U \longrightarrow X$ such that

$$\frac{\|Nx\|}{\|x\|} \longrightarrow 0 \qquad \text{for } \|x\| \to 0. \tag{2.1}$$

Assume there exists a $\lambda \in \mathbb{K} \setminus \{0\}$ and a sequence $(\lambda_k)_k \subset \mathbb{K}$ and $(x_k)_k \subset U$ with the following properties:

a) $x_k \neq 0$ for all $k \in \mathbb{N}$ and $x_k \to 0$ as $k \to \infty$.

b) $\lambda_k \neq \lambda$ for all $k \in \mathbb{N}$ and $\lambda_k \to \lambda$ as $k \to \infty$.

c) $\lambda_k x_k = Kx_k + Nx_k$ for all $k \in \mathbb{N}$.

Show that λ is an eigenvalue of K.

Hint: Assume that λ is not an eigenvalue and use theorems of Fredholm operators to obtain a suitable resolvent. Find a representation for x_k in terms of that resolvent and lead this to a contradiction.

Exercise 5.3

Let H be a separable Hilbert space and let $A \in \mathcal{L}(H)$ be a compact and self–adjoint operator. Let $a^* \geq 0$ be the biggest eigenvalue of A and $a_* \leq 0$ be the smallest eigenvalue of A.

a) Show that one of the two equalities $a^* = \|A\|$ or $a_* = -\|A\|$ holds.

b) Let $B \in \mathcal{L}(H)$ be another compact and self–adjoint operator. Let $b^* \geq 0$ be the biggest eigenvalue of B and $b_* \leq 0$ be the smallest eigenvalue of B. Let $\lambda^* \geq 0$ be the biggest eigenvalue of $A + B$ and $\lambda_* \leq 0$ be the smallest eigenvalue of $A + B$. Show that

$$\lambda^* \leq a^* + b^*, \qquad \lambda_* \geq a_* + b_*. \tag{3.1}$$

Exercise 5.4

Let $\mathcal{H} := L^2((0,1), \mathbb{C})$ and $\mathcal{D}(A) := \{f \in H^2((0,1), \mathbb{C}) : f(0) = f(1),\ f'(0) = f'(1)\}$. Let $A \colon \mathcal{D}(A) \longrightarrow \mathcal{H}$ be the periodic Laplace operator (see Example 1.21). Determine all eigenvalues and eigenfunctions of A. Do these eigenfunctions form an orthonormal basis of $\mathcal{H}$? Justify your answer.

Exercise 5.5
Let $\mathcal{H} := L^2((0,1),\mathbb{C})$. Define the operator

$$Au(x) := \int_0^x u(y)\,\mathrm{d}y \qquad \text{for all } u \in \mathcal{H}. \tag{5.1}$$

a) Show that $A\colon \mathcal{H} \longrightarrow \mathcal{H}$ is a compact operator.

b) Determine $\sigma_p(A)$ and $\sigma(A)$.

c) Is A a self–adjoint operator? Justify your answer.

Hint: You can use compact embedding theorems from the theory of Sobolev spaces.

Exercise 5.6
Let H be a separable Hilbert space and $A \in \mathcal{L}(H)$ be self–adjoint. Let $N \in \mathbb{N} \cup \{\infty\}$. Show that there exists a decomposition

$$H = \bigoplus_{n=1}^{N} H_n \tag{6.1}$$

with subspaces $H_n \subset H$ for every $n \in \{1,\dots,N\}$, such that:

a) For every $n \in \{1,\dots,N\}$ the space H_n is A invariant, i.e. for $x \in H_n$ we have $Ax \in H_n$.

b) For every $n \in \{1,\dots,N\}$ there exists $y_n \in H_n$, such that y_n is cyclic for the restriction $A|_{H_n}$, i.e.

$$H_n = \overline{\{f(A)y_n : f \in C^0(\sigma(A))\}}. \tag{6.2}$$

Hint: Use the theorem of Stone–Weierstraß and Zorn's lemma.