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Exercise	4.1	4.2	4.3	$\sum$
Points:				

Functional Analysis 2 – Exercise Sheet 4

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Exercise 4.1

Let H be a separable Hilbertspace and $A \in \mathcal{J}_1$. Show that the series $\sum_k (\varphi_k, A\varphi_k)$ converges absolutely for every orthonormal basis $(\varphi_k)_k \subset H$ and that the limit is independent of the choice of the orthonormal basis.

Hint: For the independence you can use that every operator in $A \in \mathcal{J}_1$ is decomposable into $A = A_+ - A_-$, where $A_+, A_- \geq 0$ and $A_+A_- = 0$.

Exercise 4.2

Let X and Y be Banach spaces and $A \in \mathcal{L}(X, Y)$ be a Fredholm operator. Show that the adjoint operator $A' \in \mathcal{L}(Y', X')$ is a Fredholm operator and $\operatorname{ind}(A') = -\operatorname{ind}(A)$.

Exercise 4.3

Let X, Y and Z be Banach spaces and let $B \in \mathcal{L}(X, Y)$ and $A \in \mathcal{L}(Y, Z)$. Show that if two of the three operators A, B and AB are Fredholm operators, then also the third one is a Fredholm operator and it holds

$$\operatorname{ind}(AB) = \operatorname{ind}(A) + \operatorname{ind}(B). \tag{3.1}$$

In order to do so, show the following statements:

a) Show that $\dim\ker(AB) = \dim\ker(B) + \dim(\ker(A) \cap \operatorname{ran}(B))$.

b) Decompose the spaces Y, $\ker(A)$ and $\operatorname{ran}(B)$ suitably and show, that Z decomposes in $\operatorname{ran}(AB)$, a subspace of $\operatorname{ran}(A)$ and a closed subspace $Z_0 \subset Z$.

c) Show that $\operatorname{codim}\ \operatorname{ran}(AB) = \operatorname{codim}\ \operatorname{ran}(A) + \operatorname{codim}\ \operatorname{ran}(B) - \dim\ker(A) + \dim(\operatorname{ran}(B) \cap \ker(A))$.

d) Show the closedness of the ranges.

e) Conclude the statement.

Hint: Considering b): use more than one decomposition of the space Y.

This exercise will be longer than the others since there are a lot of cases to consider, so the amount of points will be doubled for this exercise. Apart from d) and e) this is an exercise in linear algebra.