Names: ______________________
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Exercise	3.1	3.2	3.3	3.4	$\sum$
Points:					

Functional Analysis 2 – Exercise Sheet 3

Winter term 2019/20, University of Heidelberg

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Exercise 3.1

Let H be a Hilbert space and $A\colon \mathcal{D}(A) \longrightarrow H$ be a symmetric and closed operator. Show that only one of the following possibilities can occur:

$$i)\ \sigma(A) = \mathbb{C}, \qquad ii)\ \sigma(A) = \{\lambda \in \mathbb{C} : \Im(\lambda) \geq 0\},$$
$$iii)\ \sigma(A) \subset \mathbb{R}, \qquad iv)\ \sigma(A) = \{\lambda \in \mathbb{C} : \Im(\lambda) \leq 0\}.$$

Conclude that in this setting A is self-adjoint if and only if $\ker(A^* \pm \mathbf{i}\,\mathsf{id}) = \{0\}$.

Hint: Exercise 2.2 might be helpful.

Exercise 3.2

Let X and Y be Banach spaces. Let $T \in \mathcal{L}(X,Y)$ and $T' \in \mathcal{L}(Y',X')$ be its adjoint.

a) Show that $\|T\|_{\mathcal{L}(X,Y)} = \|T'\|_{\mathcal{L}(Y',X')}$.

b) Show that T is a compact operator if and only if T' is a compact operator.

Hint: Statement b) is the statement of Proposition 1.26 (*iii*). You are allowed to use the remaining statements of that proposition for this exercise.

Exercise 3.3

Let H be a Hilbert space and let $A \in \mathcal{L}(H)$ be self-adjoint. Show that

$$\|A\| = \sup_{\lambda \in \sigma(A)} |\lambda| = \sup_{\|x\|=1} |(x, Ax)|. \tag{3.1}$$

Hint: Exercise 1.4 might be helpful. Use the parallelogram law to deduce

$$|(x, Ay)| \leq \sup_{\|z\|=1} |(z, Az)|$$

for all $x, y \in H$ such that $\|x\| = \|y\| = 1$.

Exercise 3.4

For a Hilbert space H and a self-adjoint operator $A \in \mathcal{L}(H)$ recall the notation $|A| := (A^*A)^{\frac{1}{2}}$. Let $H = \mathbb{C}^2$ and define the matrices $X, Y \in \mathcal{L}(H)$ via

$$X := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Y := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.1}$$

a) Compute $|X + \mathsf{id}|$ and $|Y - \mathsf{id}|$.

b) Show that it does not hold $|(X + \mathsf{id}) + (Y - \mathsf{id})| \leq |X + \mathsf{id}| + |Y - \mathsf{id}|$.