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| Exercise | 2.1 | 2.2 | 2.3 | 2.4 | $\sum$ |
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# Functional Analysis 2 – Exercise Sheet 2

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## Exercise 2.1

Let $H$ be a separable Hilbert space and let $A\colon \mathcal{D}(A) \longrightarrow H$ and $B\colon \mathcal{D}(B) \longrightarrow H$ be two symmetric operators such that $A \subset B$. Show that $B^* \subset A^*$ and conclude that any symmetric extension of a self-adjoint operator is the operator itself.

## Exercise 2.2

Let $H$ be a Hilbert space and $A\colon \mathcal{D}(A) \longrightarrow H$ be a symmetric and closed operator. In this exercise we want to show that the function $\lambda \longmapsto \dim\ker(A^* - \lambda\,\mathsf{id})$ is constant on the half spaces $\mathbb{C}_\pm \coloneqq \{\lambda \in \mathbb{C} : \pm\Im(\lambda) > 0\}$. We will restrict ourselves to the upper halfspace $\mathbb{C}_+$, the argument for $\mathbb{C}_-$ is analogueous.

a) Let $M, N \subset H$ be closed subspaces such that $M \cap N^\perp = \{0\}$. Show that $\dim M \leq \dim N$.

b) Let $\lambda \in \mathbb{C}$ such that $\Im(\lambda) \neq 0$. Show that $\operatorname{ran}(A - \lambda\,\mathsf{id})$ is closed.

c) Let $\lambda \in \mathbb{C}_+$ and $\mu \in \mathbb{C}$ such that $|\lambda - \mu| < \Im(\lambda)$. Show that $\ker(A^* - \mu\,\mathsf{id}) \cap \ker(A^* - \lambda\,\mathsf{id})^\perp = \{0\}$.

d) Show that $\lambda \longmapsto \dim(A^* - \lambda\,\mathsf{id})$ is locally constant on $\mathbb{C}_+$ and conclude that this map is actually constant on the whole halfspace $\mathbb{C}_+$.

*Hint:* Recall the identity $\|(A - \lambda\,\mathsf{id})x\|^2 = \|(A - \Re(\lambda)\,\mathsf{id})x\|^2 + \Im(\lambda)^2\,\|x\|^2$. You can use that it holds $\operatorname{ran}(A^*)^\perp = \ker(A)$ for closed operators. In order to show c) use an indirect proof.

## Exercise 2.3

Let $\mathcal{H} \coloneqq L^2((0,1),\mathbb{C})$, $\mathcal{D}(A) \coloneqq H_0^2((0,1),\mathbb{C})$ and $\mathcal{D}(B) \coloneqq H_0^1((0,1),\mathbb{C})$. Let $\varepsilon > 0$ and define the operators $A\colon \mathcal{D}(A) \longrightarrow \mathcal{H}$ and $B\colon \mathcal{D}(B) \longrightarrow \mathcal{H}$ via

$$A \coloneqq -\Big(\frac{\mathrm{d}}{\mathrm{d}x}\Big)^2, \qquad\qquad B \coloneqq \mathbf{i}\varepsilon\,\frac{\mathrm{d}}{\mathrm{d}x}. \tag{3.1}$$

Show that $A + B\colon \mathcal{D}(A) \longrightarrow \mathcal{H}$ is a self-adjoint operator.

*Hint:* Recall Young's inequality: $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ for $\frac{1}{p} + \frac{1}{q} = 1$ and $a, b > 0$.

## Exercise 2.4

Let $H = L^2(\mathbb{R}^n,\mathbb{C})$ and $A \coloneqq \mathcal{D}(A) \longrightarrow H$ be a self-adjoint operator. Let $\psi \in C^1([0,\infty),\mathcal{D}(A))$ be a solution to the time-dependent homogeneous Schrödinger equation

$$\mathbf{i}\,\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = A\psi(t) \qquad\qquad \text{for all } t > 0 \tag{4.1}$$

with initial data $\psi(0) = \psi_0 \in \mathcal{D}(A)$. Let $\|\psi_0\| = 1$. Show that $\|\psi(t)\| = 1$ for all $t > 0$.